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On the Disturbance of the Steady Flow of an Inviscid Liquid between Parallel Planes

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II. On the Disturbance of the Steady Flow of an Inviscid Liquid between Parallel Planes

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CONTENTS

I—INTRODUCTION

1—In a number of papers dealing with the stability of fluid motion, RAYLEIGH* employed a certain method, which we may refer to as the "characteristic-value" method. For some problems this method gives results in agreement with observation. For example, it establishes that a heterogeneous inviscid liquid at rest under gravity is stable if the density decreases steadily as we pass upward†; it establishes that an inviscid liquid rotating between concentric circular cylinders is stable if, and only if, the square of the circulation increases steadily as we pass outward. This result was stated by RAYLEIGH‡, and its validity appears to be confirmed by the

* 'Proc. Lond. Math. Soc.,' vol. 11, p. 57 (1880) (Sci. Papers, vol. 1, p. 474); vol. 14, p. 170 (1883) (Sci. Papers, vol. 2, p. 200); 'Phil. Mag.,' vol. 34, p. 59 (1892) (Sci. Papers, vol. 3, p. 575); 'Phil. Mag.,' vol. 26, p. 1001 (1913) (Sci. Papers, vol. 6, p. 197); 'Phil. Mag.,' vol. 30, p. 329 (1915) (Sci. Papers, vol. 6, p. 341).

† RAYLEIGH, 'Proc. Lond. Math. Soc.,' vol. 14, p. 170 (1883) (Sci. Papers, vol. 2, p. 200).

‡ 'Proc. Roy. Soc.,' A, vol. 93, p. 148 (1917) (Sci. Papers, vol. 6, p. 447).

experiments of TAYLOR,* but a simple mathematical proof by the characteristic-value method was not given. I have recently supplied such a proof, extending the problem to include a heterogeneous liquid.†

But when the method is applied to some other problems, the situation is not so satisfactory. Among the results to which RAYLEIGH‡ was led is the following. If an inviscid liquid flows between parallel planes, the motion is stable if d^2u_0/dy^2 retains the same sign throughout the liquid, u_0 being the velocity in the steady motion and y the distance from one of the planes. This result is deduced from the fact that the characteristic values of a parameter in a certain differential equation cannot be complex, the implication being that they are therefore real. RAYLEIGH§ further claimed that the method established the stability of a uniform shearing motion, for which $d^2u_0/dy^2 = 0$. KELVIN‖ and LOVE¶ criticized the method, and a review of the situation in 1907 was given by ORR.** In spite of the fact that its general validity remains obscure, the characteristic-value method has been widely employed.†† It is not the purpose of the present paper to attempt to justify or to discredit the characteristic-value method in general. The paper deals only with the simplest of all stability problems, that of an inviscid liquid flowing between fixed parallel planes. In §2 the method is discussed in some detail and in §3 an argument is developed to show that RAYLEIGH's criterion for stability, mentioned above, cannot be legitimately deduced by his method. He proved that complex characteristic values are impossible, and I now prove that real characteristic values are also impossible. The conclusion to be drawn is that the characteristic-value method is not applicable to this case.

In one of his papers‡‡ RAYLEIGH suggested another method, based on the conservation of vorticity. This method, which may be described as "the method of vorticity," does not seem to have been developed, although it is more promising than the characteristic-value method. Part III of the present paper contains a systematic application of the method of vorticity to the problem of the disturbance of an inviscid homogeneous liquid flowing between parallel fixed planes. With a slight degree of indeterminacy, the distribution of vorticity determines the distribution of velocity: in §4 some expressions for velocity in terms of vorticity are given. In

* 'Phil. Trans.,' A, vol. 223, p. 289 (1923).

† 'Trans. Roy. Soc. Canada,' Sec. III, vol. 27, p. 1 (1933).

‡ 'Proc. Lond. Math. Soc.,' vol. 11, p. 57 (1880) (Sci. Papers, vol. 1, p. 474).

§ 'Proc. Lond. Math. Soc.,' vol. 11, p. 69 (1880) (Sci. Papers, vol. 1, p. 486); 'Phil. Mag.,' vol. 26, p. 1003 (1913) (Sci. Papers, vol. 6, p. 198).

‖ 'Nature,' vol. 23, p. 45 (1880) (Math. and Phys. Papers, vol. 4, p. 186); 'Phil. Mag.,' vol. 24, p. 272 (1887) (Math. and Phys. Papers, vol. 4, p. 330).

¶ 'Proc. Lond. Math. Soc.,' vol. 27, p. 199 (1896).

** 'Proc. Roy. Irish Acad.,' vol. 27A, p. 9 (1907).

†† *Cf.* SOMMERFELD, 'Atti Cong. matemat. Roma,' vol. 3, p. 116 (1908); HOPF, 'Ann. Physique,' vol. 44, p. 1 (1914); TAYLOR, 'Phil. Trans.,' A, vol. 223, p. 289 (1923); SEXL, 'Ann. Physique,' vol. 83, p. 835 (1927), vol. 84, p. 807 (1927). For a bibliography of the stability problem, see BATEMAN, 'Bull. Nat. Res. Coun. Wash.,' vol. 84, p. 382 (1932).

‡‡ 'Phil. Mag.,' vol. 26, p. 1001 (1913) (Sci. Papers, vol. 6, p. 197).

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§5 a method is outlined for the determination of the effect of a finite disturbance by a process of successive approximations involving integro-differential equations. This process is not, however, employed in the subsequent part of the paper, the argument being there confined to the first approximation, when we have to deal with a single integro-differential equation for the vorticity. By developing the vorticity as a Fourier series in y (measured across the stream) y is eliminated in §6 ; x (measured along the stream) is eliminated in §7, on the assumption that the initial disturbance is periodic in x and represented by a finite number of sinusoidal terms, and the problem of the determination of the distribution of vorticity at time t is reduced to the solution of an infinite system of ordinary differential equations of the first order for an infinite set of functions of t, whose values for $t = 0$ are assigned with the initial disturbance. There is a noteworthy difference between the characteristic-value method and the method of vorticity : in the former the boundary conditions on the walls (vanishing normal velocity) remain in evidence and constitute the essence of the characteristic-value problem, whereas in the method of vorticity these boundary conditions are automatically satisfied from the very beginning of the investigation by the method of images.

In §8 it is shown that the infinite set of ordinary differential equations, to whose solution the problem has been reduced, admit solutions in the form of power series in t which converge for all values of t, provided that in the steady motion the velocities of slipping on the two walls are equal. In §9 it is shown how the coefficients in these power series may be calculated, and reference is made to the parabolic velocity-profile. The process of calculation is complicated and no actual calculations are carried out for two reasons. In the first place, the most interesting general question is that of stability and the most interesting profile the parabolic. However, I have been able to establish the stability of this profile by a simple and different method, so that there is no point in attempting calculations along the lines of the present theoretical solution. Secondly, the question of an inviscid liquid is of comparatively little physical interest, and although the present paper contains the solution of a classical problem, it may be of more importance in suggesting, for the investigation of the disturbance of a viscous liquid, a method more valid than that of characteristic values.

The question of the linear velocity-profile is not touched on. It is quite a special case, and of little interest. The vorticity of the undisturbed motion is constant and the vorticity of a small disturbance is carried along with the velocity of the undisturbed stream. It hardly seems that anything further need be said of it.

II—The Characteristic-Value Method

2—*Review of the characteristic-value method*

Let there be an inviscid liquid of uniform density ρ, flowing in the direction of the x-axis between parallel planes $y = \pm b$, the velocity-components being

$$u = u_0 (y), \qquad v = 0. \qquad (2.1)$$

On this motion there is superimposed at time $t = 0$ a distribution of small velocities. (We shall confine our attention to two-dimensional disturbances). Denoting by u', v', p' the excesses of the velocity-components and pressure in the resulting motion over the corresponding quantities in the steady motion, and neglecting the second and higher powers of u', v', p' and their derivatives, the equations of motion take the approximate form

$$\left.\begin{aligned} \frac{\partial u'}{\partial t} + u_0 \frac{\partial u'}{\partial x} + v \frac{du_0}{dy} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x} \\ \frac{\partial v'}{\partial t} + u_0 \frac{\partial v'}{\partial x} &= -\frac{1}{\rho}\frac{\partial p'}{\partial y} \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.2)$$

We seek to obtain solutions of these partial differential equations for u', v', p' satisfying the following boundary conditions :—

u' and v' shall equal assigned functions of x and y for $t = 0$ . . (2.3)

$$v' = 0 \text{ for } y = \pm b, \quad t > 0. \quad . \quad . \quad . \quad . \quad (2.4)$$

The assigned values of u', v' for $t = 0$ are, of course, such as to satisfy the last of (2.2) and also (2.4).

Let us assume that the initial disturbance is periodic in x with period $2a$. Leaving aside for the moment the explicit expression of this initial disturbance, we seek solutions of (2.2) of the type

$$u' = U_r(y)\, e^{i\alpha_r(x-ct)}, \quad v' = V_r(y)\, e^{i\alpha_r(x-ct)}, \quad p' = P_r(y)\, e^{i\alpha_r(x-ct)}, \quad . \quad . \quad (2.5)$$

where $\alpha_r = \pi r/a$, r being an integer (not zero), and c is a constant. Substitution gives

$$\left.\begin{aligned} i\alpha_r (u_0 - c)\, U_r + \frac{du_0}{dy} V_r &= -\, i\alpha_r \frac{P_r}{\rho} \\ i\alpha_r (u_0 - c)\, V_r &= -\frac{1}{\rho}\frac{dP_r}{dy} \\ i\alpha_r U_r + \frac{dV_r}{dy} &= 0 \end{aligned}\right\}, \quad . \quad . \quad . \quad (2.6)$$

so that (2.5) satisfy (2.2) provided that V_r satisfies the differential equation*

$$(u_0 - c)\frac{d^2V_r}{dy^2} - \left\{\alpha_r^2 (u_0 - c) + \frac{d^2u_0}{dy^2}\right\} V_r = 0, \quad . \quad . \quad . \quad (2.7)$$

and U_r, P_r are given by

$$U_r = \frac{i}{\alpha_r}\frac{dV_r}{dy}, \quad P_r = \frac{i\rho}{\alpha_r}\frac{du_0}{dy} V_r - (u_0 - c)\frac{i\rho}{\alpha_r}\frac{dV_r}{dy}. \quad . \quad . \quad . \quad (2.8)$$

* Rayleigh, 'Proc. Lond. Math. Soc.,' vol. 11, p. 57 (1880) (Sci. Papers, vol. 1, p. 474), equation (51).

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Moreover, the boundary conditions (2.4) will be satisfied provided that

$$V_r(b) = V_r(-b) = 0. \qquad (2.9)$$

The solution of (2.7) with the boundary conditions (2.9) constitutes a characteristic-value problem, and we may expect that a solution (real or complex) will exist if and only if c has one of a set of characteristic values (real or complex). Assuming that these characteristic values exist, let us denote them (since they depend on r) by

$$c_{r1}, \quad c_{r2}, \quad \ldots, \qquad (2.10)$$

and let us denote the corresponding characteristic functions by

$$V_{r1}, \quad V_{r2}, \quad \ldots. \qquad (2.11)$$

If we build up infinite linear combinations from (2.5) of the form

$$\left.\begin{aligned} u' &= \sum_{r=-\infty}^{\infty} (r \neq 0) \sum_{s=1}^{\infty} A_{rs} U_{rs}(y)\, e^{i\alpha_r(x-c_{rs}t)} \\ v' &= \sum_{r=-\infty}^{\infty} (r \neq 0) \sum_{s=1}^{\infty} A_{rs} V_{rs}(y)\, e^{i\alpha_r(x-c_{rs}t)} \\ p' &= \sum_{r=-\infty}^{\infty} (r \neq 0) \sum_{s=1}^{\infty} A_{rs} P_{rs}(y)\, e^{i\alpha_r(x-c_{rs}t)} \end{aligned}\right\}, \qquad (2.12)$$

where

$$U_{rs} = \frac{i}{\alpha_r}\frac{dV_{rs}}{dy}, \quad P_{rs} = \frac{i\rho}{\alpha_r}\frac{du_0}{dy}V_{rs} - (u_0 - c)\frac{i\rho}{\alpha_r}\frac{dV_{rs}}{dy}, \qquad (2.13)$$

and A_{rs} are constants, arbitrary for the present, then these expressions constitute formal solutions of (2.2) with the boundary conditions (2.4), and they will be actual solutions provided that the required term-by-term differentiations of the infinite series are permissible.

Turning now to the other boundary conditions (2.3), it appears from (2.12) that the constants A_{rs} are to be chosen so that

$$\left.\begin{aligned} (u')_{t=0} &= \sum_{r=-\infty}^{\infty} (r \neq 0) \sum_{s=1}^{\infty} A_{rs} U_{rs}(y)\, e^{i\alpha_r x} \\ (v')_{t=0} &= \sum_{r=-\infty}^{\infty} (r \neq 0) \sum_{s=1}^{\infty} A_{rs} V_{rs}(y)\, e^{i\alpha_r x} \end{aligned}\right\}, \qquad (2.14)$$

the left hand sides being the assigned initial velocities of disturbance. Since these initial velocities are assumed to be periodic in x, it will be possible to expand $(v')_{t=0}$ in the form

$$(v')_{t=0} = \sum_{r=-\infty}^{\infty} \phi_r(y)\, e^{i\alpha_r x}, \qquad (2.15)$$

where, by virtue of (2.4),

$$\phi_r(b) = \phi_r(-b) = 0. \qquad (2.16)$$

The last of (2.2) then gives

$$\left(\frac{\partial u'}{\partial x}\right)_{t=0} = -\sum_{r=-\infty}^{\infty} \frac{d\phi_r}{dy} e^{i\alpha_r x}, \quad \ldots \ldots \ldots \quad (2.17)$$

and hence

$$(u')_{t=0} = i \sum_{r=-\infty}^{\infty} (r \neq 0) \frac{1}{\alpha_r} \frac{d\phi_r}{dy} e^{i\alpha_r x} - x \frac{d\phi_0}{dy} + \mathrm{F}(y). \quad \ldots \quad (2.18)$$

Since, by hypothesis, this is periodic in x, we must have $d\phi_0/dy = 0$, and therefore, by (2.16), $\phi_0 = 0$. Thus the equation of continuity demands that the term for $r = 0$ be absent from the expansion (2.15). Moreover, the arbitrary function F (y) in (2.18) may be absorbed into u_0, and hence we may regard the initial velocity of disturbance as given by

$$\left.\begin{aligned} (u')_{t=0} &= i \sum_{r=-\infty}^{\infty} (r \neq 0) \frac{1}{\alpha_r} \frac{d\phi_r}{dy} e^{i\alpha_r x} \\ (v')_{t=0} &= \sum_{r=-\infty}^{\infty} (r \neq 0)\, \phi_r(y)\, e^{i\alpha_r x} \end{aligned}\right\} \quad \ldots \quad (2.19)$$

In order to make the expressions for $(v')_{t=0}$ in (2.14) and (2.19) agree, it is merely necessary to expand $\phi_r(y)$ in terms of the characteristic functions $\mathrm{V}_{rs}(y)$ and choose A_{rs} equal to the coefficients in the expansion, so that

$$\phi_r(y) = \sum_{s=1}^{\infty} \mathrm{A}_{rs} \mathrm{V}_{rs}(y). \quad \ldots \ldots \ldots \quad (2.20)$$

Then assuming that term-by-term differentiation is permissible, we have by (2.13)

$$\frac{i}{\alpha_r} \frac{d\phi_r}{dy} = \frac{i}{\alpha_r} \sum_{s=1}^{\infty} \mathrm{A}_{rs} \frac{d\mathrm{V}_{rs}}{dy} = \sum_{s=1}^{\infty} \mathrm{A}_{rs} \mathrm{U}_{rs}, \quad \ldots \ldots \quad (2.21)$$

and thus the expressions for $(u')_{t=0}$ in (2.14) and (2.19) agree. Thus if the characteristic values c_{rs} exist, and if the expansions and term-by-term processes (differentiation and proceeding to the limit $t = 0$) are permissible, we have in (2.12) expressions which give the disturbance at any time t. If all the characteristic values are real, it appears probable from (2.12) (and is accepted without question in applications of the characteristic-value method) that u', v' remain small, and hence that the given motion is stable. If, on the other hand, there are complex characteristic values, there will be an exponential instability, because complex characteristic values will occur in conjugate pairs. Hence the question of stability is reduced to an examination of the reality of the characteristic values of (2.7). That is the essence of the characteristic-value method.

The preceding argument has been given at some length, because comparatively little attention has been directed to the infinite processes involved in the characteristic-value method. Also, for the purposes of §3, it is desirable to point out the necessity for continuity in the characteristic functions *and their first derivatives*. If we take an initial disturbance such that $(u')_{t=0}$ and $(v')_{t=0}$ are continuous in x and y

(and it is this type of disturbance which we have had in mind), then by (2.19) $\phi_r(y)$ and $d\phi_r/dy$ will be continuous; then in order that (2.20) and (2.21) may hold, it is necessary that not only the characteristic functions V_{rs}, but also their derivatives dV_{rs}/dy, should be continuous. This point is important, because it is just this continuity of the derivatives which leads to the results of §3. This necessity for the continuity of the first derivatives was not admitted by ORR*, but it certainly appears to be demanded if we assume the continuity of $(u')_{t=0}$; if we do not assume this latter continuity, the whole treatment must be different. The critical situation arises when we suppose that in the characteristic equation (2.7) c has a real characteristic value, equal to some value which u_0 takes in the liquid, let us say for $y = y_1$. If we divide (2.7) by $u_0 - c$, obtaining

$$\frac{d^2V_r}{dy^2} - \left\{\alpha_r^2 + \frac{d^2u_0/dy^2}{u_0 - c}\right\} V_r = 0, \quad \ldots \ldots \ldots \quad (2.22)$$

the coefficient of V_r becomes infinite for $y = y_1$, provided that d^2u_0/dy^2 does not vanish there. This is the "disturbing infinity" which troubled KELVIN (*loc. cit.*). RAYLEIGH† however held that it sufficed to have the function V_r continuous everywhere, including such critical points, but that it was not necessary to have dV_r/dy continuous at such a point. For reasons already stated, this view does not appear to be tenable, and we shall assume that dV_r/dy must be continuous everywhere.

3—*A criticism of* RAYLEIGH'S *condition for stability*

RAYLEIGH‡ showed that if d^2u_0/dy^2 does not vanish in the range of y, then the differential equation (2.7), with the boundary conditions (2.9), cannot be satisfied if c has a complex value. Hence, in accordance with the general method outlined above, the motion is stable provided that real characteristic values exist and the necessary infinite processes are legitimate. We refer to the non-vanishing of d^2u_0/dy^2 as "RAYLEIGH'S condition for stability." We shall now show that no real characteristic values exist.

Let us first assume that c has a real value outside the range of values of u_0, and let us put (writing V, α instead of V_r, α_r)

$$\chi = \frac{V}{u_0 - c}. \quad \ldots \ldots \ldots \ldots \quad (3.1)$$

Then χ is regular throughout the range $-b \leqslant y \leqslant b$, and in terms of χ the differential equation (2.7) becomes

$$\frac{d}{dy}\left\{(u_0 - c)^2 \frac{d\chi}{dy}\right\} - \alpha^2 (u_0 - c)^2 \chi = 0. \quad \ldots \ldots \quad (3.2)$$

* *loc. cit.*, p. 22.

† 'Phil. Mag.,' vol. 26, p. 1001 (1913) (Sci. Papers, vol. 6, p. 197; in particular pp. 198-199).

‡ 'Proc. Lond. Math. Soc.,' vol. 11, p. 57 (1880) (Sci. Papers, vol. 1, p. 474).

Multiplying by $\chi\, dy$ and integrating from $-b$ to b, we obtain

$$\left[(u_0 - c)^2\, \chi \frac{d\chi}{dy}\right]_{-b}^{b} - \int_{-b}^{b} (u_0 - c)^2 \left(\frac{d\chi}{dy}\right)^2 dy - \alpha^2 \int_{-b}^{b} (u_0 - c)^2\, \chi^2\, dy = 0. \qquad (3.3)$$

But χ vanishes for $y = \pm b$, by virtue of (2.9), and thus the first term in (3.3) vanishes. The resulting equation is satisfied only by the trivial $\chi = 0$. Hence there is no real characteristic value outside the range of values of u_0.

Let us now assume that there is a real characteristic value c, lying in the range of values of u_0 and corresponding to $y = y_1$. Let us assume that d^2u_0/dy^2 does not vanish in the liquid. We shall first investigate the solutions of (2.7) in the neighbourhood of $y = y_1$, putting

$$y - y_1 = z. \qquad (3.4)$$

We have

$$u_0 - c = u'_0\, z + \tfrac{1}{2} u''_0\, z^2 + \ldots, \qquad (3.5)$$

where u'_0, u''_0 are constants. Equation (2.7) becomes

$$(u'_0 z + \tfrac{1}{2}u''_0 z^2 + \ldots)\frac{d^2V}{dz^2} - \{\alpha^2(u'_0 z + \tfrac{1}{2}u''_0 z^2 + \ldots) + u''_0 + \ldots\}\, V = 0. \qquad (3.6)$$

Substituting

$$V = z^m\, (a_0 + a_1 z + \ldots), \qquad (3.7)$$

we obtain as indicial equation, if $u'_0 \neq 0$,

$$m\,(m-1) = 0. \qquad (3.8)$$

But $m = 0$ is impossible, since $u''_0 \neq 0$. Hence $m = 1$, and we have for V a power series starting with a term in z. The other solution is*

$$V = (z \log z)\ (a_0 + a_1 z + \ldots) + \text{power series in } z\,; \qquad (3.9)$$

it is inadmissible, since dV/dz is not finite for $z = 0$. Thus near $y = y_1$, V is of the order of z, and if we again define χ by (3.1), χ is regular throughout the range $-b \leqslant y \leqslant b$.

If $u'_0 = 0$, the indicial equation becomes

$$m^2 - m - 2 = 0, \qquad (3.10)$$

and we must take $m = 2$; the other solution of (3.6) is again inadmissible, and χ, again defined as above, is regular throughout the range.

We then proceed exactly as in (3.2) and (3.3). If y_1 is not a terminal point of the range $(-b, b)$, the first term in (3.3) vanishes since χ vanishes at $y = \pm b$. On the other hand, if y_1 is one of the ends of the range, then χ and $d\chi/dy$ are finite there,

* *Cf.* LOVE, *loc. cit.*, p. 208.

and the contribution to the first term in (3.3) vanishes by virtue of the vanishing of $u_0 - c$. Hence, on the assumption that d^2u_0/dy^2 does not vanish in the liquid, there can exist no real characteristic value c for the equation (2.7). Nor, as RAYLEIGH showed, can there exist a complex characteristic value. Hence the characteristic-value method is inadmissible in this case.

III—The Method of Vorticity

4—*Expressions for velocity-components in terms of vorticity*

Before proceeding to determine by the method of vorticity the effect of an initial disturbance on an assigned steady motion, we shall first develop some expressions for the components of velocity in a liquid confined between parallel planes $y = \pm b$, the vorticity being assigned at the instant in question.

THEOREM I—*If a liquid moves irrotationally between the fixed planes $y = \pm b$, except for the presence of a point-vortex of strength κ at (x_1, y_1), and if the liquid is at rest at infinity, then the components of velocity at any point (x, y) are given by*

$$u - iv = \frac{i\kappa}{8b}\left(\tanh\frac{\pi(z-\bar{z}_1)}{4b} - \coth\frac{\pi(z-z_1)}{4b}\right), \quad \dots \quad (4.1)$$

where

$$z = x + iy, \qquad z_1 = x_1 + iy_1, \qquad \bar{z}_1 = x_1 - iy_1. \quad \dots \quad (4.2)$$

This is a known result, or immediately deducible from known results*, since the system of images of the vortex in the planes $y = \pm b$ form two collinear rows of vortices of strengths κ and $-\kappa$. It is, however, easy to verify directly that (4.1) gives a motion satisfying the following conditions :—

(i) *The motion is without expansion and irrotational except for $z = z_1$* : this follows from the fact that $u - iv$ is an analytic function of z for any point z between the planes $y = \pm b$ except $z = z_1$.

(ii) *The circulation in a circuit surrounding z_1 is κ* : this is seen from the fact that for such a circuit

$$\int (u - iv)\,(dx + idy) = \kappa. \quad \dots \quad (4.3)$$

(iii) $v = 0$ *for* $y = \pm b$: this follows from the fact that in general

$$\tanh\frac{\pi(z-\bar{z}_1)}{4b} - \coth\frac{\pi(z-z_1)}{4b} = -\frac{2\cos\frac{\pi y_1}{2b}}{\sinh\frac{\pi}{2b}(x+iy-x_1) - i\sin\frac{\pi y_1}{2b}}, \quad (4.4)$$

and when $y = \pm b$ this becomes

$$-\frac{2\cos\frac{\pi y_1}{2b}}{\pm i\cosh\frac{\pi(x-x_1)}{2b} - i\sin\frac{\pi y_1}{2b}}, \quad \dots \quad (4.5)$$

a pure imaginary.

* LAMB, "Hydrodynamics," p. 224 (1932).

(iv) $u \to 0$, $v \to 0$ *as* $|x| \to \infty$: in fact if

$$|x - x_1| > \frac{2b}{\pi} \log_e 4, \qquad (4.6)$$

we have

$$\left| \tanh \frac{\pi (z - \bar{z}_1)}{4b} - \coth \frac{\pi (z - z_1)}{4b} \right| < 8 \exp\left(-\frac{\pi |x - x_1|}{2b}\right), \qquad (4.7)$$

and so, as $|x| \to \infty$, u and v tend to zero as $\exp(-\pi |x| / 2b)$.

THEOREM II—*If in the motion of a liquid between the fixed planes* $y = \pm b$, *the vorticity is* $\omega (x, y, t)$, *where as usual*

$$\omega = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \qquad (4.8)$$

then the components of velocity at any point (x, y) *are given by*

$$u - iv = \chi (t) + \frac{i}{4b} \iint \omega (x_1, y_1, t) \left(\tanh \frac{\pi (z - \bar{z}_1)}{4b} - \coth \frac{\pi (z - z_1)}{4b}\right) dx_1\, dy_1, \qquad (4.9)$$

where $\chi (t)$ *is real and undetermined, and the integral is taken over the region between the planes. If, for a certain* t, $\omega (x, y, t) = 0$ *outside a finite range of* x, *or if*

$$\lim_{|x| \to \infty} \omega (x, y, t) = 0, \qquad (4.10)$$

then $\chi (t)$ *is the velocity at infinity.*

While the validity of (4.9) is indicated by the way in which it is constructed from (4.1) by putting $2\omega\, dx_1\, dy_1$ instead of κ, and integrating, to prove the theorem rigorously we have to establish the following points (i), (ii), (iii):

(i) *The integral in* (4.9) *converges* (*provided that* ω *is bounded*). The convergence at infinity is obvious from (4.7). We have also to consider convergence at poles of the integrand, which can occur only where

$$\frac{\pi (z - \bar{z}_1)}{4b} = (2n + 1) \frac{\pi i}{2}, \quad \text{or} \quad \frac{\pi (z - z_1)}{4b} = n\pi i, \qquad (4.11)$$

where n is zero or an integer. Since z, z_1, $\bar{z}_1$ are confined to the strip $-b \leqslant y \leqslant b$, the only singularity is at $z_1 = z$; this is a pole of $\coth \{\pi (z - z_1)/4b\}$, and also of $\tanh \{\pi (z - \bar{z}_1)/4b\}$ if z lies on $y = \pm b$. Now

$$\coth \frac{\pi (z - z_1)}{4b} = \frac{4b}{\pi (z - z_1)} + f(z - z_1), \qquad (4.12)$$

where f tends to zero as z_1 approaches z. Thus the question of the convergence of

$$\iint \omega (x_1, y_1, t) \coth \frac{\pi (z - z_1)}{4b}\, dx_1\, dy_1$$

reduces to that of the convergence of

$$\iint \omega\,(x_1, y_1, t)\,\frac{1}{z_1 - z}\,dx_1\,dy_1\,;$$

in polar co-ordinates r, θ, having z for pole, this last integral becomes

$$\iint \omega\,(x_1, y_1, t)\,e^{-\theta i}\,dr\,d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.13)$$

which obviously converges. If z lies on $y = \pm b$, then

$$\tanh\frac{\pi\,(z - \bar{z}_1)}{4b} = \tanh\frac{\pi\,(\bar{z} - \bar{z}_1 \pm 2ib)}{4b} = \coth\frac{\pi\,(\bar{z} - \bar{z}_1)}{4b}, \quad . \quad (4.14)$$

and the convergence of

$$\iint \omega\,(x_1, y_1, t)\,\tanh\frac{\pi\,(z - \bar{z}_1)}{4b}\,dx_1\,dy_1$$

follows as above. Thus the convergence of the integral in (4.9) is established.

(ii) *Equation* (4.9) *defines a motion without expansion and with a vorticity* ω. For any point z in the liquid (not on a wall) we have

$$\tanh\frac{\pi\,(z - \bar{z}_1)}{4b} - \coth\frac{\pi\,(z - z_1)}{4b} = -\frac{4b}{\pi\,(z - z_1)} + \frac{4b}{i}\,\mathrm{F}\,(z, x_1, y_1), \quad (4.15)$$

where, for all values of x_1, y_1, F is analytic in z, without poles. Thus (4.9) may be written

$$u - iv = \chi\,(t) - \frac{i}{\pi}\iint \omega\,(x_1, y_1, t)\,\frac{1}{z - z_1}\,dx_1\,dy_1$$
$$+ \iint \omega\,(x_1, y_1, t)\,\mathrm{F}\,(z, x_1, y_1)\,dx_1\,dy_1. \quad (4.16)$$

We may differentiate under the sign of integration in the second integral, and so obtain

$$\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)\iint \omega\,(x_1, y_1, t)\,\mathrm{F}\,(z, x_1, y_1)\,dx_1\,dy_1$$
$$= \iint \omega\,(x_1, y_1, t)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)\mathrm{F}\,(z, x_1, y_1)\,dx_1\,dy_1$$
$$= 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.17)$$

Moreover,

$$\iint \omega\,(x_1, y_1, t)\,\frac{1}{z - z_1}\,dx_1\,dy_1 = \iint \omega\,(x_1, y_1, t)\,\frac{x - x_1}{r^2}\,dx_1\,dy_1$$
$$- i\iint \omega\,(x_1, y_1, t)\,\frac{y - y_1}{r^2}\,dx_1\,dy_1, \quad (4.18)$$

where $r = |z - z_1|$. But if we have a two-dimensional distribution of attractive matter of density ω, the logarithmic potential at (x, y) is

$$V(x, y, t) = \iint \omega(x_1, y_1, t) \log \frac{1}{r} \, dx_1 \, dy_1, \quad . \quad . \quad . \quad . \quad (4.19)$$

and the components of attraction are

$$\frac{\partial V}{\partial x} = \iint \omega(x_1, y_1, t) \frac{x_1 - x}{r^2} \, dx_1 \, dy_1, \qquad \frac{\partial V}{\partial y} = \iint \omega(x_1, y_1, t) \frac{y_1 - y}{r^2} \, dx_1 \, dy_1. \quad (4.20)$$

Also

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -2\pi\omega(x, y, t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.21)$$

We have then

$$\iint \omega(x_1, y_1, t) \frac{1}{z - z_1} \, dx_1 \, dy_1 = -\frac{\partial V}{\partial x} + i \frac{\partial V}{\partial y}, \quad . \quad . \quad . \quad . \quad (4.22)$$

and so, by (4.16) and (4.17),

$$\begin{aligned} \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)(u - iv) &= -\frac{i}{\pi}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)\left(-\frac{\partial V}{\partial x} + i\frac{\partial V}{\partial y}\right) \\ &= \frac{i}{\pi}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \\ &= -2i\omega(x, y, t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.23) \end{aligned}$$

Therefore

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega(x, y, t), \quad . \quad . \quad . \quad . \quad . \quad (4.24)$$

as it was required to prove.

(iii) $v = 0$ *for* $y = \pm b$; this follows at once from (4.5), since the integrand in (4.9) is a pure imaginary if $y = \pm b$.

We have seen that the velocity distribution given by (4.9) has the required vorticity and satisfies the boundary conditions on $y = \pm b$. A motion satisfying these conditions is undetermined only for the possible superposition of an irrotational motion satisfying the boundary conditions on $y = \pm b$. But the most general irrotational motion of this type is a translational velocity parallel to the x-axis: thus theorem II is established, except for the last sentence. To establish that, we have to show that under the stated condition the integral in (4.9) tends to zero as $|x|$ tends to infinity. Let x be large and positive; let us divide the liquid into three regions R_1, R_2, R_3, such that for a variable point (x_1, y_1)

$$\text{in } R_1,\ x_1 < x/2\,; \quad \text{in } R_2,\ x/2 < x_1 < 3x/2\,; \quad \text{in } R_3,\ 3x/2 < x_1. \qquad (4.25)$$

If ϖ is the maximum of $|\omega(x_1, y_1, t)|$ in the liquid, then by (4.7) the contribution to (4.9) from R_1 is less in absolute value than

$$4\varpi \int_{-\infty}^{x/2} \exp\left(-\frac{\pi(x - x_1)}{2b}\right) dx_1 = \frac{8b\varpi}{\pi} \exp\left(-\frac{\pi x}{4b}\right). \quad . \quad . \quad . \quad . \quad (4.26)$$

The contribution from R_3 is less in absolute value than this same expression, which tends to zero as x tends to infinity. As to the contribution from R_2, let ω' be the maximum of $|\omega(x_1, y_1, t)|$ in this region. Now it follows from (4.7) that

$$\iint \left| \tanh \frac{\pi(z-\bar{z}_1)}{4b} - \coth \frac{\pi(z-z_1)}{4b} \right| dx_1\, dy_1 < K, \quad . \quad . \quad . \quad (4.27)$$

where the integral is taken throughout the whole liquid, and K is some constant. Hence the contribution to (4.9) from R_2 is less in absolute value than $K\omega'/4b$. But if $\omega(x, y, t)$ vanishes outside a finite region or if the condition (4.10) is fulfilled, this contribution is zero or tends to zero as $x \to \infty$. The case where $x \to -\infty$ is similarly treated. Thus as $|x| \to \infty$, the integral in (4.9) tends to zero, and theorem II is established.

It follows immediately that if there is superimposed on the motion

$$u = u_0(y), \qquad v = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.28)$$

a motion with vorticity ω', then the velocity-components at (x, y) are given by

$$u - iv = \chi(t) + u_0(y) + \frac{i}{4b} \iint \omega'(x_1, y_1, t) \left(\tanh \frac{\pi(z-\bar{z}_1)}{4b} - \coth \frac{\pi(z-z_1)}{4b} \right) dx_1\, dy_1, \quad . \quad . \quad . \quad (4.29)$$

where $\chi(t)$ is real and undetermined.

In very general circumstances, the vorticity $\omega(x, y, t)$ may be expanded for the range $(-b, b)$ of y in a Fourier series of the form

$$\omega(x, y, t) = \sum_{r=-\infty}^{\infty} \Phi_r(x, t) \exp(r\pi y i/b). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.30)$$

We shall now establish the following result:

THEOREM III—*If, in a liquid moving between the fixed planes $y = \pm b$, the vorticity is given by* (4.30), *then the components of velocity du, dv at a point x, y due to the vorticity in the strip $x = x_1$, $x = x_1 + dx_1$ (and its images) are given by the formulæ*:

for $x_1 < x$,

$$du - i\,dv = i\,dx_1 \sum_{p=1}^{\infty} \{\Phi_p(x_1, t) - \Phi_{-p}(x_1, t)\} \exp\{p\pi(x_1 - z)/b\}$$
$$- \frac{4i\,dx_1}{\pi} \sum_{r=-\infty}^{\infty} \sum_{p=0}^{\infty} \Phi_r(x_1, t) \exp \frac{(2p+1)\pi(x_1 - z)}{2b} \cdot \frac{(-1)^{r+p}(2p+1)}{(2p+1)^2 - 4r^2}, \quad (4.31)$$

for $x_1 > x$,

$$du - i\,dv = i\,dx_1 \sum_{p=1}^{\infty} \{\Phi_p(x_1, t) - \Phi_{-p}(x_1, t)\} \exp\{p\pi(z - x_1)/b\}$$
$$+ \frac{4i dx_1}{\pi} \sum_{r=-\infty}^{\infty} \sum_{p=0}^{\infty} \Phi_r(x_1, t) \exp \frac{(2p+1)\pi(z - x_1)}{2b} \cdot \frac{(-1)^{r+p}(2p+1)}{(2p+1)^2 - 4r^2}. \quad (4.32)$$

To establish this result, we have by (4.9), for the components of velocity due to the strip in question,

$$du - i\,dv = \frac{idx_1}{4b}\int_{-b}^{b}\omega\,(x_1, y_1, t)\left(\tanh\frac{\pi\,(z-\bar{z}_1)}{4b} - \coth\frac{\pi\,(z-z_1)}{4b}\right)dy_1. \quad (4.33)$$

We now substitute for $\omega\,(x_1\ y_1, t)$ the series (4.30) with x_1, y_1 instead of x, y, and for the other part of the integral one or other of the expressions

for $x_1 < x$,

$$\tanh\frac{\pi\,(z-\bar{z}_1)}{4b} - \coth\frac{\pi\,(z-z_1)}{4b}$$
$$= 2\sum_{m=1}^{\infty}\exp\left(-\frac{m\pi z}{2b}\right)\left\{(-1)^m\exp\frac{m\pi\bar{z}_1}{2b} - \exp\frac{m\pi z_1}{2b}\right\}, \quad . \quad . \quad (4.34)$$

for $x_1 > x$,

$$\tanh\frac{\pi\,(z-\bar{z}_1)}{4b} - \coth\frac{\pi\,(z-z_1)}{4b}$$
$$= 2\sum_{m=1}^{\infty}\exp\frac{m\pi z}{2b}\left\{-(-1)^m\exp\left(-\frac{m\pi\bar{z}_1}{2b}\right) + \exp\left(-\frac{m\pi z_1}{2b}\right)\right\}. \quad (4.35)$$

Thus we obtain

for $x_1 < x$,

$$du - idv = \frac{idx_1}{2b}\int_{-b}^{b}\sum_{r=-\infty}^{\infty}\sum_{m=1}^{\infty}\Phi_r\,(x_1, t)\exp\frac{r\pi y_1 i}{b}\exp\left(-\frac{m\pi z}{2b}\right)$$
$$\times\left\{(-1)^m\exp\frac{m\pi\bar{z}_1}{2b} - \exp\frac{m\pi z_1}{2b}\right\}dy_1$$
$$= \frac{idx_1}{2b}\sum_{r=-\infty}^{\infty}\sum_{m=1}^{\infty}\Phi_r\,(x_1, t)\exp\frac{m\pi\,(x_1-z)}{2b}\,\{(-1)^m I_{r,-m} - I_{r,m}\}, \quad . \quad (4.36)$$

for $x_1 > x$,

$$du - idv = \frac{idx_1}{2b}\int_{-b}^{b}\sum_{r=-\infty}^{\infty}\sum_{m=1}^{\infty}\Phi_r\,(x_1, t)\exp\frac{r\pi y_1 i}{b}\exp\frac{m\pi z}{2b}$$
$$\times\left\{-(-1)^m\exp\left(-\frac{m\pi\bar{z}_1}{2b}\right) + \exp\left(-\frac{m\pi z_1}{2b}\right)\right\}dy_1$$
$$= \frac{idx_1}{2b}\sum_{r=-\infty}^{\infty}\sum_{m=1}^{\infty}\Phi_r\,(x_1, t)\exp\frac{m\pi\,(z-x_1)}{2b}\,\{-(-1)^m I_{r,m} + I_{r,-m}\}, \quad (4.37)$$

where

$$I_{r\,m} = \int_{-b}^{b}\exp\{(r+\tfrac{1}{2}m)\,\pi y_1 i/b\}\,dy_1$$
$$= \begin{cases} 2b \text{ if } 2r+m=0 \\ \dfrac{4b\,(-1)^r\sin\frac{1}{2}m\pi}{\pi\,(2r+m)} \quad \text{if} \quad 2r+m \neq 0 \end{cases} \quad (r=0, \pm 1, \ldots;\ m = \pm 1, \pm 2, \ldots). \quad (4.38)$$

Dividing the summations in (4.36), (4.37) into summations for m even and m odd, we see that for any even value of m ($m = 2p$) there are surviving terms only when $r = \pm p$. Hence (4.31), (4.32) follow immediately.

Adding together the expressions for the velocity-components due to all the strips which make up the whole liquid, and adding the arbitrary function as in (4.9), we establish the following result directly by integration of (4.31), (4.32),

THEOREM IV—*If the vorticity in a liquid moving between the fixed planes $y = \pm b$ is given by* (4.30), *then the components of velocity at any point x, y are given by*

$$u - iv = \chi(t) + \sum_{p=1}^{\infty} M_p(x, t) \exp(-p\pi yi/b) + \sum_{p=1}^{\infty} M'_p(x, t) \exp(p\pi yi/b)$$
$$- \frac{4i}{\pi} \sum_{r=-\infty}^{\infty} \sum_{p=0}^{\infty} L_{r,p}(x, t) \exp\left(-\frac{(2p+1)\pi yi}{2b}\right) \cdot \frac{(-1)^{r+p}(2p+1)}{(2p+1)^2 - 4r^2}$$
$$+ \frac{4i}{\pi} \sum_{r=-\infty}^{\infty} \sum_{p=0}^{\infty} L'_{r,p}(x, t) \exp\frac{(2p+1)\pi yi}{2b} \cdot \frac{(-1)^{r+p}(2p+1)}{(2p+1)^2 - 4r^2}, \quad (4.39)$$

where

$$\left.\begin{aligned} L_{r,p}(x, t) &= \int_{-\infty}^{x} \exp\frac{(2p+1)\pi(x_1 - x)}{2b} \Phi_r(x_1, t)\, dx_1 \\ L'_{r,p}(x, t) &= \int_{x}^{\infty} \exp\frac{(2p+1)\pi(x - x_1)}{2b} \Phi_r(x_1, t)\, dx_1 \\ M_p(x, t) &= i\int_{-\infty}^{x} \exp\frac{p\pi(x_1 - x)}{b} \{\Phi_p(x_1, t) - \Phi_{-p}(x_1, t)\}\, dx_1 \\ M'_p(x, t) &= i\int_{x}^{\infty} \exp\frac{p\pi(x - x_1)}{b} \{\Phi_p(x_1, t) - \Phi_{-p}(x_1, t)\}\, dx_1 \end{aligned}\right\}, \quad (4.40)$$

$$(r = 0, \pm 1, \ldots; \; p = 0, 1, 2, \ldots),$$

and $\chi(t)$ is an arbitrary real function.

Since ω, as given by (4.30), is a real quantity, the quantities

$$\Phi_r + \Phi_{-r}, \quad i(\Phi_r - \Phi_{-r})$$

are real. Hence, picking out the imaginary parts of (4.39), we have

$$v = \sum_{p=1}^{\infty} \{M_p(x, t) - M_{-p}(x, t)\} \sin(p\pi y/b)$$
$$+ \frac{4}{\pi} \sum_{r=-\infty}^{\infty} \sum_{p=0}^{\infty} \{L_{r,p}(x, t) - L'_{r,p}(x, t)\} \cos\frac{(2p+1)\pi y}{2b} \frac{(-1)^{r+p}(2p+1)}{(2p+1)^2 - 4r^2}. \quad (4.41)$$

5—*Finite disturbance: reduction of the problem to the solution of a system of integro-differential equations*

Let us now consider how the preceding results are to be applied to the problem of stability. For any two-dimensional motion of an inviscid liquid, we have the equation which expresses the conservation of vorticity:

$$\frac{d\omega}{dt} \equiv \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0. \qquad (5.1)$$

Let us consider the motion to be a superposition on an undisturbed flow $u = u_0(y)$, $v = 0$, for which the vorticity is

$$\omega_0 = -\tfrac{1}{2}\frac{du_0}{dy}, \qquad (5.2)$$

of a disturbance for which the velocity has components u', v' at time t and for which the vorticity is ω'. If we assume the disturbance to be small and neglect second and higher powers of small quantities, (5.1) takes the approximate form

$$\frac{\partial \omega'}{\partial t} + u_0\frac{\partial \omega'}{\partial x} - \tfrac{1}{2}\frac{d^2u_0}{dy^2}v' = 0, \qquad (5.3)$$

a familiar equation. If we write

$$\tanh\frac{\pi(z-\bar{z}_1)}{4b} - \coth\frac{\pi(z-z_1)}{4b} = \mathrm{R}(x, y, x_1, y_1) + i\mathrm{S}(x, y, x_1, y_1), \qquad (5.4)$$

where R and S are real, (5.3) may be written (*cf.* (4.9))

$$\left(\frac{\partial}{\partial t} + u_0(y)\frac{\partial}{\partial x}\right)\omega'(x, y, t)$$
$$+\frac{1}{8b}\frac{d^2u_0}{dy^2}\iint \omega'(x_1, y_1, t)\,\mathrm{R}(x, y, x_1, y_1)\,dx_1\,dy_1 = 0. \qquad (5.5)$$

This is an integro-differential equation for the determination of $\omega'(x, y, t)$, the values of $\omega'(x, y, 0)$ being assigned. When ω' has been determined, the velocity at x, y, t is given by

$$u - iv = \chi(t) + u_0(y) + \frac{i}{4b}\iint \omega'(x_1, y_1, t)(\mathrm{R} + i\mathrm{S})\,dx_1\,dy_1, \qquad (5.6)$$

where $\chi(t)$ is a real function, undetermined in the absence of further information.

To push the investigation beyond the first approximation, we may proceed by writing for the velocity-components of the disturbance and the corresponding vorticity

$$\left.\begin{aligned} u' &= u'_{(1)} + u'_{(2)} + \dots \\ v' &= v'_{(1)} + v'_{(2)} + \dots \\ \omega' &= \omega'_{(1)} + \omega'_{(2)} + \dots \\ \omega'_{(r)} &= \tfrac{1}{2}\left(\frac{\partial v'_{(r)}}{\partial x} - \frac{\partial u'_{(r)}}{\partial y}\right) \end{aligned}\right\}, \quad \dots \quad (5.7)$$

the subscripts indicating order. Substitution in the exact equation (5.1) gives

$$\frac{\partial}{\partial t}(\omega'_{(1)} + \omega'_{(2)} + \dots) + (u_0 + u'_{(1)} + u'_{(2)} + \dots)\frac{\partial}{\partial x}(\omega'_{(1)} + \omega'_{(2)} + \dots)$$
$$+ (v'_{(1)} + v'_{(2)} + \dots)\left(-\tfrac{1}{2}\frac{d^2u_0}{dy^2} + \frac{\partial\omega'_{(1)}}{\partial y} + \dots\right) = 0, \quad . \quad . \quad (5.8)$$

from which we obtain formulæ for successive approximations :

$$\frac{\partial\omega'_{(1)}}{\partial t} + u_0\frac{\partial\omega'_{(1)}}{\partial x} - \tfrac{1}{2}\frac{d^2u_0}{dy^2}v'_{(1)} = 0, \quad \dots \quad (5.9)$$

$$\frac{\partial\omega'_{(2)}}{\partial t} + u_0\frac{\partial\omega'_{(2)}}{\partial x} - \tfrac{1}{2}\frac{d^2u_0}{dy^2}v'_{(2)} = -u'_{(1)}\frac{\partial\omega'_{(1)}}{\partial x} - v'_{(1)}\frac{\partial\omega'_{(1)}}{\partial y}, \quad \dots \quad (5.10)$$

and generally

$$\left.\begin{aligned} &\frac{\partial\omega'_{(n)}}{\partial t} + u_0\frac{\partial\omega'_{(n)}}{\partial x} - \tfrac{1}{2}\frac{d^2u_0}{dy^2}v'_{(n)} = f_{(n)}(x, y, t) \\ &f_{(n)} = -\sum_{r=1}^{n-1}\left((u'_{(r)}\frac{\partial\omega'_{(n-r)}}{\partial x} + v'_{(r)}\frac{\partial\omega'_{(n-r)}}{\partial y}\right), \quad (n = 1, 2, \dots) \end{aligned}\right\} . \quad . \quad . \quad (5.11)$$

These equations form an integro-differential system when we substitute for $u'_{(n)}$, $v'_{(n)}$ from

$$u'_{(n)} - iv'_{(n)} = \chi_{(n)}(t) + \frac{i}{4b}\iint\omega'_{(n)}(x_1, y_1, t)\,(\mathrm{R} + i\mathrm{S})\,dx_1\,dy_1, \quad . \quad . \quad (5.12)$$

where $\chi_{(n)}(t)$ are real undetermined functions. This indeterminacy does not present itself in calculating the vorticity to the first approximation, since only the imaginary part of the expression is required (*cf.* (5·5)). If the disturbance is "localized" in the sense of theorem II, the functions $\chi_{(n)}(t)$ will vanish. Otherwise the removal of the indeterminacy requires some further data, such as the value of u' at some special point for all values of t.

With the equations (5.11) are to be associated the initial conditions

$$\omega'_{(1)}(x, y, 0) = \omega'(x, y, 0), \quad \ldots \ldots \quad (5.13)$$

this being the initial vorticity of the disturbance, and

$$\omega'_{(n)}(x, y, 0) = 0, \qquad (n = 2, 3, \ldots). \quad \ldots \ldots \quad (5.14)$$

6—*Small disturbance: elimination of y*

Let us consider the first approximation, for which the vorticity of the disturbance satisfies (5.3). Let the vorticity of the disturbance be expanded in the form

$$\omega'(x, y, t) = \sum_{r=-\infty}^{\infty} \Phi_r(x, t) \exp(r\pi yi/b). \qquad (6.1)$$

The problem of determining the motion, when the initial disturbance is given, consists in finding $\Phi_r(x, t)$ when $\Phi_r(x, 0)$ are assigned.

Let us now expand the several terms of (5.3) in Fourier series of the above type. We have

$$\frac{\partial \omega'}{\partial t} = \sum_{r=-\infty}^{\infty} \frac{\partial \Phi_r}{\partial t} \exp(r\pi yi/b), \quad \ldots \ldots \quad (6.2)$$

and

$$u_0 \frac{\partial \omega'}{\partial x} = \sum_{r=-\infty}^{\infty} H_r(x, t) \exp(r\pi yi/b), \quad \ldots \ldots \quad (6.3)$$

where

$$H_r(x, t) = \frac{1}{2b} \int_{-b}^{b} u_0(y) \sum_{s=-\infty}^{\infty} \frac{\partial \Phi_s}{\partial x} \exp\{(s - r)\pi yi/b\}\, dy$$

$$= \sum_{s=-\infty}^{\infty} A_{r,s} \frac{\partial \Phi_s}{\partial x}, \quad \ldots \ldots \quad (6.4)$$

where $A_{r,s}$ are constants,

$$A_{r,s} = \frac{1}{2b} \int_{-b}^{b} u_0(y) \exp\{(s - r)\pi yi/b\}\, dy. \quad \ldots \ldots \quad (6.5)$$

As for the last term in (5.3), v' is given, without change of notation, by the series written in (4.41). Thus

$$-\tfrac{1}{2} \frac{d^2 u_0}{dy^2} v' = \sum_{r=-\infty}^{\infty} J_r(x, t) \exp(r\pi yi/b), \quad \ldots \ldots \quad (6.6)$$

where

$$J_r(x, t) = -\frac{1}{4b} \int_{-b}^{b} \frac{d^2 u_0}{dy^2} v' \exp(-r\pi yi/b)\, dy$$

$$= -\tfrac{1}{2} \sum_{p=1}^{\infty} B_{r,p} \{M_p(x, t) - M_{-p}(x, t)\}$$

$$- \frac{2}{\pi} \sum_{s=-\infty}^{\infty} \sum_{p=0}^{\infty} C_{r,p} \{L_{s,p}(x, t) - L'_{s,p}(x, t)\} \frac{(-1)^{s+p}(2p+1)}{(2p+1)^2 - 4s^2}, \quad (6.7)$$

where $B_{r,p}$, $C_{r,p}$ are constants,

$$\left.\begin{aligned} B_{r,p} &= \frac{1}{2b}\int_{-b}^{b}\frac{d^2u_0}{dy^2}\exp\left(-\frac{r\pi yi}{b}\right)\sin\left(\frac{p\pi y}{b}\right)dy \\ C_{r,p} &= \frac{1}{2b}\int_{-b}^{b}\frac{d^2u_0}{dy^2}\exp\left(-\frac{r\pi yi}{b}\right)\cos\frac{(2p+1)\pi y}{2b}\,dy \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6.8)$$

By (6.2), (6.3), (6.6), the several terms of (5.3) are now expressed as Fourier series. The coefficient of $\exp(r\pi yi/b)$ is then to be equated to zero, which gives

$$\frac{\partial\Phi_r}{\partial t} + H_r(x,t) + J_r(x,t) = 0, \qquad (r = 0, \pm 1, \pm 2, \ldots)\,; \qquad (6.9)$$

hence we may state this result :

THEOREM V—*When the steady motion* $u = u_0(y)$, $v = 0$, *between the fixed planes* $y = \pm b$, *is slightly disturbed, and the vorticity of the disturbance is expressed in the form* (6.1), *the functions* $\Phi_r(x, t)$ *satisfy the infinite system of integro-differential equations*

$$\frac{\partial\Phi_r}{\partial t} + \sum_{s=-\infty}^{\infty} A_{r,s}\frac{\partial\Phi_s}{\partial x} - \tfrac{1}{2}\sum_{p=1}^{\infty} B_{r,p}\{M_p(x,t) - M'_p(x,t)\}$$

$$-\frac{2}{\pi}\sum_{s=-\infty}^{\infty}\sum_{p=0}^{\infty} C_{r,p}\{L_{s,p}(x,t) - L'_{s,p}(x,t)\}\frac{(-1)^{s+p}(2p+1)}{(2p+1)^2-4s^2} = 0, \qquad (6.10)$$

$$(r = 0, \pm 1, \pm 2, \ldots),$$

where M_p, M'_p, $L_{s,p}$, $L'_{s,p}$ *are functions of* x, t, *obtained from the* Φ's *by integration according to the formulae* (4.40), *and* $A_{r,p}$, $B_{r,p}$, $C_{r,p}$ *are constants, given by* (6.5), (6.8) *and depending only on the velocity-profile of the undisturbed motion.*

If u_0, d^2u_0/dy^2 are expanded in Fourier series,

$$u_0 = \sum_{q=-\infty}^{\infty} U_q \exp(q\pi yi/b), \qquad d^2u_0/dy^2 = \sum_{q=-\infty}^{\infty} V_q \exp(q\pi yi/b), \quad . \quad . \quad (6.11)$$

we find from (6.5), (6.8)

$$\left.\begin{aligned} A_{r,s} &= U_{r-s}, \\ B_{r,p} &= \frac{i}{2}(V_{r+p} - V_{r-p}), \\ C_{r,p} &= \frac{2}{\pi}\sum_{m=-\infty}^{\infty} V_{m+r}\frac{(-1)^{p+m}(2p+1)}{(2p+1)^2-4m^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6.12)$$

We shall now write (6.10) in another form, which is in general less compact, but which is of interest for the cases where u_0 is either an odd or an even function of y. Let us write (6.1) in the equivalent form

$$\omega'(x, y, t) = \tfrac{1}{2}F_0(x, t) + \sum_{r=1}^{\infty} F_r(x, t)\cos(r\pi y/b) + \sum_{r=1}^{\infty} G_r(x, t)\sin(r\pi y/b), \qquad (6.13)$$

so that F_r, G_r are real, and are connected with the complex Φ_r by the relations

$$\left\{\begin{array}{ll} F_r = \Phi_r + \Phi_{-r}, & \Phi_r = \tfrac{1}{2}(F_r - i\,G_r), \\ G_r = i(\Phi_r - \Phi_{-r}), & \Phi_{-r} = \tfrac{1}{2}(F \;+ i\,G_r), \end{array}\right\} \; (r = 0, 1, 2, \ldots;\; G_0 = 0). \qquad (6.14)$$

Let us change the sign of r in (6.10), and add the result to (6.10): this gives

$$\frac{\partial F_r}{\partial t} + \tfrac{1}{2}(A_{r,0} + A_{-r,0})\frac{\partial F_0}{\partial x} + \tfrac{1}{2}\sum_{s=1}^{\infty}(A_{r,s} + A_{-r,s} + A_{r,-s} + A_{-r,-s})\frac{\partial F_s}{\partial x}$$
$$- \frac{i}{2}\sum_{s=1}^{\infty}(A_{r,s} + A_{-r,s} - A_{r,-s} - A_{-r,-s})\frac{\partial G_s}{\partial x}$$
$$- \tfrac{1}{2}\sum_{p=1}^{\infty}(B_{r,p} + B_{-r,p})\{M_p(x, t) - M'_p(x, t)\}$$
$$- \frac{2}{\pi}\left[\sum_{p=0}^{\infty}(C_{r,p} + C_{-r,p})\{L_{0,p}(x, t) - L'_{0,p}(x, t)\}\frac{(-1)^p}{2p+1}\right.$$
$$+ \sum_{s=1}^{\infty}\sum_{p=0}^{\infty}(C_{r,p} + C_{-r,p})\{L_{s,p}(x, t) + L_{-s,p}(x, t) - L'_{s,p}(x, t) - L'_{-s,p}(x, t)\}$$
$$\left.\times \frac{(-1)^{s+p}(2p+1)}{(2p+1)^2 - 4s^2}\right] = 0, \qquad (r = 0, 1, 2, \ldots); \quad \ldots \ldots \ldots \quad (6.15)$$

on the other hand, if we multiply (6.10) by i, change the sign of r and subtract, we get

$$\frac{\partial G_r}{\partial t} + \frac{i}{2}(A_{r,0} - A_{-r,0})\frac{\partial F_0}{\partial x} + \frac{i}{2}\sum_{s=1}^{\infty}(A_{r,s} - A_{-r,s} + A_{r,-s} - A_{-r,-s})\frac{\partial F_s}{\partial x}$$
$$+ \tfrac{1}{2}\sum_{s=1}^{\infty}(A_{r,s} - A_{-r,s} - A_{r,-s} + A_{-r,-s})\frac{\partial G_s}{\partial x}$$
$$- \frac{i}{2}\sum_{p=1}^{\infty}(B_{r,p} - B_{-r,p})\{M_p(x, t) - M'_p(x, t)\}$$
$$- \frac{2i}{\pi}\left[\sum_{p=0}^{\infty}(C_{r,p} - C_{-r,p})\{L_{0,p}(x, t) - L'_{0,p}(x, t)\}\frac{(-1)^p}{2p+1}\right.$$
$$+ \sum_{s=1}^{\infty}\sum_{p=0}^{\infty}(C_{r,p} - C_{-r,p})\{L_{s,p}(x, t) + L_{-s,p}(x, t) - L'_{s,p}(x, t) - L'_{-s,p}(x, t)\}$$
$$\left.\times \frac{(-1)^{s+p}(2p+1)}{(2p+1)^2 - 4s^2}\right] = 0, \qquad (r = 1, 2, \ldots) \quad \ldots \ldots \ldots \quad (6.16)$$

If we adopt the notation

$$\left.\begin{aligned}
A'_{r,s} &= \tfrac{1}{4}(A_{r,s}+A_{-r,s}+A_{r,-s}+A_{-r,-s}) = \frac{1}{2b}\int_{-b}^{b} u_0(y)\cos\frac{r\pi y}{b}\cos\frac{s\pi y}{b}\,dy\\
A''_{r,s} &= -\frac{i}{4}(A_{r,s}+A_{-r,s}-A_{r,-s}-A_{-r,-s}) = \frac{1}{2b}\int_{-b}^{b} u_0(y)\cos\frac{r\pi y}{b}\sin\frac{s\pi y}{b}\,dy\\
A'''_{r,s} &= \frac{i}{4}(A_{r,s}-A_{-r,s}+A_{r,-s}-A_{-r,-s}) = \frac{1}{2b}\int_{-b}^{b} u_0(y)\sin\frac{r\pi y}{b}\cos\frac{s\pi y}{b}\,dy\\
A''''_{r,s} &= \tfrac{1}{4}(A_{r,s}-A_{-r,s}-A_{r,-s}+A_{-r,-s}) = \frac{1}{2b}\int_{-b}^{b} u_0(y)\sin\frac{r\pi y}{b}\sin\frac{s\pi y}{b}\,dy\\
B'_{r,p} &= \tfrac{1}{2}(B_{r,p}+B_{-r,p}) = \frac{1}{2b}\int_{-b}^{b}\frac{d^2u_0}{dy^2}\cos\frac{r\pi y}{b}\sin\frac{p\pi y}{b}\,dy\\
B''_{r,p} &= \frac{i}{2}(B_{r,p}-B_{-r,p}) = \frac{1}{2b}\int_{-b}^{b}\frac{d^2u_0}{dy^2}\sin\frac{r\pi y}{b}\sin\frac{p\pi y}{b}\,dy\\
C'_{r,p} &= \tfrac{1}{2}(C_{r,p}+C_{-r,p}) = \frac{1}{2b}\int_{-b}^{b}\frac{d^2u_0}{dy^2}\cos\frac{r\pi y}{b}\cos\frac{(2p+1)\pi y}{2b}\,dy\\
C''_{r,p} &= \frac{i}{2}(C_{r,p}-C_{-r,p}) = \frac{1}{2b}\int_{-b}^{b}\frac{d^2u_0}{dy^2}\sin\frac{r\pi y}{b}\cos\frac{(2p+1)\pi y}{2b}\,dy
\end{aligned}\right\}, \quad (6.17)$$

all of which are real constants, depending only on the velocity-profile of the undisturbed motion, and if we put

$$\left.\begin{aligned}
L''_{s,p}(x,t) &= L_{s,p}(x,t)+L_{-s,p}(x,t)-L'_{s,p}(x,t)-L'_{-s,p}(x,t)\\
&= \int_{-\infty}^{x}\exp\frac{(2p+1)\pi(x_1-x)}{2b}F_s(x_1,t)\,dx_1\\
&\qquad -\int_{x}^{\infty}\exp\frac{(2p+1)\pi(x-x_1)}{2b}F_s(x_1,t)\,dx_1\\
M''_p(x,t) &= M_p(x,t)-M'_p(x,t)\\
&= \int_{-\infty}^{x}\exp\frac{p\pi(x_1-x)}{b}G_p(x_1,t)\,dx_1\\
&\qquad -\int_{x}^{\infty}\exp\frac{p\pi(x-x_1)}{b}G_p(x_1,t)\,dx_1
\end{aligned}\right\}, \quad (6.18)$$

then (6.15), (6.16) may be written

$$\frac{\partial F_r}{\partial t}+A'_{r,0}\frac{\partial F_0}{\partial x}+2\sum_{s=1}^{\infty}A'_{r,s}\frac{\partial F_s}{\partial x}+2\sum_{s=1}^{\infty}A''_{r,s}\frac{\partial G_s}{\partial x}-\sum_{p=1}^{\infty}B'_{r,p}M''_p(x,t)$$
$$-\frac{2}{\pi}\left[\sum_{p=0}^{\infty}C'_{r,p}L''_{0,p}(x,t)\frac{(-1)^p}{2p+1}+2\sum_{s=1}^{\infty}\sum_{p=0}^{\infty}C'_{r,p}L''_{s,p}(x,t)\frac{(-1)^{s+p}(2p+1)}{(2p+1)^2-4s^2}\right]=0, \quad (6.19)$$
$$(r=0,1,2,\ldots),$$

$$\frac{\partial G_r}{\partial t}+A'''_{r,0}\frac{\partial F_0}{\partial x}+2\sum_{s=1}^{\infty}A'''_{r,s}\frac{\partial F_s}{\partial x}+2\sum_{s=1}^{\infty}A''''_{r,s}\frac{\partial G_s}{\partial x}-\sum_{p=1}^{\infty}B''_{r,p}M''_p(x,t)$$
$$-\frac{2}{\pi}\left[\sum_{p=0}^{\infty}C''_{r,p}L''_{0,p}(x,t)\frac{(-1)^p}{2p+1}+2\sum_{s=1}^{\infty}\sum_{p=0}^{\infty}C''_{r,p}L''_{s,p}(x,t)\frac{(-1)^{s+p}(2p+1)}{(2p+1)^2-4s^2}\right]=0, \quad (6.20)$$
$$(r=1,2,\ldots).$$

These are in fact the integro-differential equations (6.10) reduced to real form. We observe that $L''_{s,p}$ is a function of the F-functions only, and M''_p a function of the G-functions only.

Let us now consider the case where u_0 is an odd function of y; d^2u_0/dy^2 is then also odd, and we have by (6.17)

$$A'_{r,s} = A''''_{r,s} = B''_{r,p} = C'_{r,p} = 0, \quad \ldots \ldots \quad (6.21)$$

and thus (6.19), (6.20) reduce to

[u_0 an odd function of y]

$$\left.\begin{aligned} &\frac{\partial F_r}{\partial t} + 2\sum_{s=1}^{\infty} A''_{r,s}\frac{\partial G_s}{\partial x} - \sum_{p=1}^{\infty} B'_{r,p} M''_p(x,t) = 0, \qquad (r = 0, 1, 2, \ldots) \\ &\frac{\partial G_r}{\partial t} + A'''_{r,0}\frac{\partial F_0}{\partial x} + 2\sum_{s=1}^{\infty} A'''_{r,s}\frac{\partial F_s}{\partial x} \\ &\quad - \frac{2}{\pi}\left[\sum_{p=0}^{\infty} C''_{r,p} L''_{0,p}(x,t)\frac{(-1)^p}{2p+1} + 2\sum_{s=1}^{\infty}\sum_{p=0}^{\infty} C''_{r,p} L''_{s,p}(x,t)\frac{(-1)^{s+p}(2p+1)}{(2p+1)^2-4s^2}\right] = 0 \\ &\qquad (r = 1, 2, \ldots) \end{aligned}\right\} . \quad \ldots \quad (6.22)$$

On the other hand, *if u_0 is an even function of y*, then we have by (6.17)

$$A''_{r,s} = A'''_{r,s} = B'_{r,p} = C''_{r,p} = 0, \quad \ldots \ldots \quad (6.23)$$

and (6.19), (6.20) reduce to

[u_0 an even function of y]

$$\left.\begin{aligned} &\frac{\partial F_r}{\partial t} + A'_{r,0}\frac{\partial F_0}{\partial x} + 2\sum_{s=1}^{\infty} A'_{r,s}\frac{\partial F_s}{\partial x} \\ &\quad - \frac{2}{\pi}\left[\sum_{p=0}^{\infty} C'_{r,p} L''_{0,p}(x,t)\frac{(-1)^p}{2p+1} + 2\sum_{s=1}^{\infty}\sum_{p=0}^{\infty} C'_{r,p} L''_{s,p}(x,t)\frac{(-1)^{s+p}(2p+1)}{(2p+1)^2-4s^2}\right] = 0 \\ &\qquad (r = 0, 1, 2, \ldots) \\ &\frac{\partial G_r}{\partial t} + 2\sum_{s=1}^{\infty} A''''_{r,s}\frac{\partial G_s}{\partial x} - \sum_{p=1}^{\infty} B''_{r,p} M''_p(x,t) = 0, \ (r = 1, 2, \ldots) \end{aligned}\right\} . \quad \ldots \quad (6.24)$$

It is remarkable that the F-functions are separated from the G-functions in these equations. Since

$$F_r(x,t) = 0, \qquad (r = 0, 1, 2, \ldots), \quad \ldots \ldots \quad (6.25)$$

means that ω' is an odd function of y, and

$$G_r(x,t) = 0, \qquad (r = 1, 2, \ldots), \quad \ldots \ldots \quad (6.26)$$

means that ω' is an even function of y, and since (6.24) imply that if either (6.25) or (6.26) is satisfied for $t = 0$, it is satisfied for all values of t, we may state the following result :

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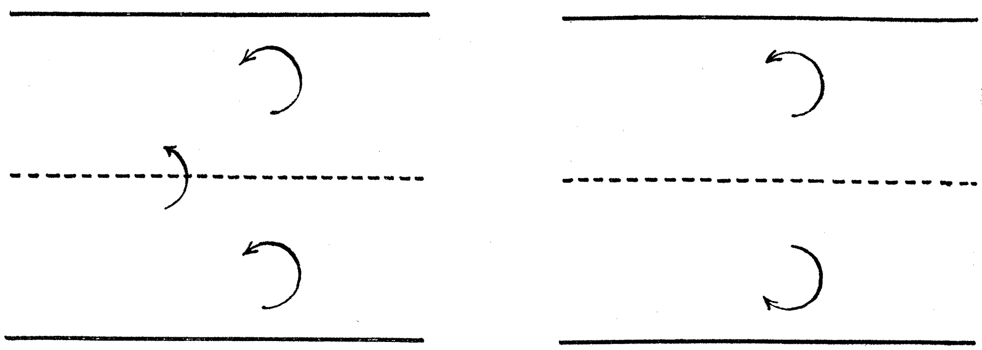
THEOREM VI—*When the steady motion* $u = u_0(y)$, $v = 0$, *between the fixed planes* $y = \pm b$, *is slightly disturbed, and the vorticity of the disturbance is expressed in the form* (6.13), *the functions* $F_r(x, t)$, $G_r(x, t)$ *satisfy the infinite system of integro-differential equations* (6.19), (6.20), *which reduce to* (6.22) *if* u_0 *is an odd function of* y *and to* (6.24) *if* u_0 *is an even function of* y. *If* u_0 *is an even function of* y *and if the vorticity of the disturbance is initially an even function of* y, *it remains an even function of* y, *whereas, if it is initially an odd function of* y, *it remains an odd function of* y; *provided always that the disturbance remains small.*

The fact that if $u_0(y)$ is even in y and $\omega'(x, y, 0)$ is odd in y, then $\omega'(x, y, t)$ is odd in y, is of course obvious, since these initial conditions correspond to a motion initially symmetrical with respect to the x-axis, and hence permanently symmetrical. The other result, where $u_0(y)$ is even in y and $\omega'(x, y, 0)$ is even in y, is unexpected. The significance of even and odd vorticities is illustrated in fig. 1.



Even vorticity Odd vorticity

Fig. 1.

7—*Small disturbance: elimination of x when the initial disturbance is periodic in x*

The disturbance considered in §6 was small, but otherwise general. The argument eliminated y, and replaced the partial differential equation (5.3) for $\omega'(x, y, t)$ by the infinite system of integro-differential equations (6.10) for the functions $\Phi_r(x, t)$. Let us now assume that the vorticity of the initial disturbance is a periodic function of x, with period $2a$, and let it be expanded in the double Fourier series

$$\omega'(x, y, 0) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \alpha_{r,s} \exp(r\pi y i/b) \exp(s\pi r i/a), \quad . \quad . \quad . \quad (7.1)$$

where $\alpha_{r,s}$ are constants determined by the initial disturbance. The assumption that $\omega'(x, y, 0)$ is continuous in x will ensure that, for large $|s|$, $|\alpha_{r,s}|$ is of the order of $1/s^2$, while the assumption that $\partial\omega'/\partial x$ is continuous will ensure that $|\alpha_{r,s}|$ is of the order of $1/|s|^3$, and so on. It does not appear, however, in the later work that any assumption of this type will necessarily suffice to establish the convergence of the series which we shall employ for the vorticity at time t. We shall therefore restrict ourselves to a *sinusoidal disturbance*, so that in (7.1) s takes only a finite range of values. We shall, however, retain the notation of (7.1), understanding that

$$\alpha_{r,s} = 0 \qquad \text{if } |s| > S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.1a)$$

Let us now put $t = 0$ in (6.1), which represents the vorticity of disturbance at any time, and compare the result with (7.1). This gives

$$\Phi_r(x, 0) = \sum_{s=-\infty}^{\infty} \alpha_{r,s} \exp(s\pi xi/a), \qquad (r = 0, \pm 1, \ldots). \quad (7.2)$$

These are the initial conditions to be satisfied in conjunction with the integro-differential equations (6.10): we shall seek solutions of the form

$$\Phi_r(x, t) = \sum_{s=-\infty}^{\infty} \omega_{r,s}(t) \exp(s\pi xi/a), \qquad (r = 0, \pm 1, \ldots), \quad (7.3)$$

where

$$\omega_{r,s}(0) = \alpha_{r,s}, \qquad (r, s = 0, \pm 1, \ldots). \quad (7.4)$$

When we substitute from (7.3) in (4.40) we obtain

$$\left.\begin{aligned} \mathrm{L}_{r,p}(x, t) - \mathrm{L}'_{r,p}(x, t) &= \int_{-\infty}^{x} \exp\frac{(2p+1)\pi(x_1 - x)}{2b} \sum_{s=-\infty}^{\infty} \omega_{r,s}(t) \exp(s\pi x_1 i/a)\, dx_1 \\ &\quad - \int_{x}^{\infty} \exp\frac{(2p+1)\pi(x - x_1)}{2b} \sum_{s=-\infty}^{\infty} \omega_{r,s}(t) \exp(s\pi x_1 i/a)\, dx_1 \\ &= -\frac{8ib}{\pi} \sum_{s=-\infty}^{\infty} \omega_{r,s}(t) \frac{\lambda_s \exp(s\pi xi/a)}{(2p+1)^2 + 4\lambda_s^2} \\ \mathrm{M}_p(x, t) - \mathrm{M}'_p(x, t) &= \frac{2b}{\pi} \sum_{s=-\infty}^{\infty} \{\omega_{p,s}(t) - \omega_{-p,s}(t)\} \frac{\lambda_s \exp(s\pi xi/a)}{p^2 + \lambda_s^2} \\ &(r = 0, \pm 1, \ldots;\ p = 0, 1, 2, \ldots) \end{aligned}\right\}, \quad (7.5)$$

where

$$\lambda_s = sb/a, \quad (7.6)$$

this being the ratio of the width of the stream ($2b$) to the wave-length of the s-disturbance ($2a/s$). Let us now substitute from (7.3), (7.5) in (6.10), and equate to zero the coefficient of $\exp(s\pi xi/a)$. This gives

$$\begin{aligned} &\frac{d\omega_{r,s}}{dt} + \frac{s\pi i}{a} \sum_{q=-\infty}^{\infty} \mathrm{A}_{r,q}\, \omega_{q,s}(t) - \frac{b}{\pi} \sum_{p=1}^{\infty} \mathrm{B}_{r,p} \frac{\lambda_s}{p^2 + \lambda_s^2} \{\omega_{p,s}(t) - \omega_{-p,s}(t)\} \\ &\quad + \frac{16ib}{\pi^2} \sum_{q=-\infty}^{\infty} \sum_{p=0}^{\infty} \mathrm{C}_{r,p}\, \omega_{q,s}(t) \frac{\lambda_s}{(2p+1)^2 + 4\lambda_s^2} \frac{(-1)^{q+p}(2p+1)}{(2p+1)^2 - 4q^2} = 0, \quad (7.7) \\ &\qquad (r, s = 0, \pm 1, \ldots), \end{aligned}$$

which may be written, since $\mathrm{B}_{r,p} = -\mathrm{B}_{r,-p}$,

$$\begin{aligned} &\frac{d\omega_{r,s}}{dt} + \lambda_s \sum_{q=-\infty}^{\infty} \omega_{q,s}(t) \left[\frac{\pi i}{b} \mathrm{A}_{r,q} - \frac{b}{\pi} \mathrm{B}_{r,q} \frac{1}{q^2 + \lambda_s^2} \right. \\ &\quad \left. + \frac{16bi}{\pi^2} \sum_{p=0}^{\infty} \mathrm{C}_{r,p} \frac{(-1)^{q+p}(2p+1)}{\{(2p+1)^2 + 4\lambda_s^2\}\{(2p+1)^2 - 4q^2\}}\right] = 0, \quad (7.8) \\ &\qquad (r, s = 0, \pm 1, \ldots). \end{aligned}$$

Hence we may state this result:

THEOREM VII—*If the vorticity of the small disturbance is initially periodic in x, and represented by (7.1) with (7.1a), then the vorticity of the disturbance at any time t is represented by the double series*

$$\omega'(x,y,t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \omega_{r,s}(t) \exp(r\pi yi/b) \exp(s\pi xi/a), \quad . \quad . \quad . \quad (7.9)$$

where the functions $\omega_{r,s}(t)$ satisfy the infinite system of ordinary differential equations of the first order

$$\frac{b}{\pi i}\frac{d\omega_{r,s}}{dt} + \lambda_s \sum_{q=-\infty}^{\infty} N_{r,q,s}\,\omega_{q,s}(t) = 0, \qquad (r, s = 0, \pm 1, \dots;\ \lambda_s = sb/a), \quad (7.10)$$

with the initial conditions (7.4), $N_{r,q,s}$ being constants which depend only on the undisturbed velocity-profile and are given by

$$\left.\begin{aligned} N_{r,q,s} &= A_{r,q} + \frac{ib^2}{\pi^2} B_{r,q} \frac{1}{q^2+\lambda_s^2} + \frac{16b^2}{\pi^3} \sum_{p=0}^{\infty} C_{r,p} \frac{(-1)^{q+p}(2p+1)}{\{(2p+1)^2+4\lambda_s^2\}\{(2p+1)^2-4q^2\}} \\ A_{r,q} &= \frac{1}{2b}\int_{-b}^{b} u_0(y) \exp\{(q-r)\pi yi/b\}\,dy \\ B_{r,q} &= \frac{1}{2b}\int_{-b}^{b} \frac{d^2u_0}{dy^2} \exp(-r\pi yi/b) \sin(q\pi y/b)\,dy \\ C_{r,p} &= \frac{1}{2b}\int_{-b}^{b} \frac{d^2u_0}{dy^2} \exp(-r\pi yi/b) \cos\{(2p+1)\pi y/2b\}\,dy \\ &\qquad (r, q, s = 0, \pm 1, \pm 2, \dots;\ p = 0, 1, 2, \dots) \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.11)$$

If we substitute in terms of the Fourier coefficients from (6.12), we have

$$N_{r,q,s} = U_{q-r} - \frac{b^2}{2\pi^2}(V_{r+q} - V_{r-q})\frac{1}{q^2+\lambda_s^2} + \frac{32b^2}{\pi^4}\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty} V_{r+m} \frac{(-1)^{q+m}(2p+1)^2}{\{(2p+1)^2+4\lambda_s^2\}\{(2p+1)^2-4q^2\}\{(2p+1)^2-4m^2\}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.12)$$

We shall revert to this expression in §8, and carry out the summation with respect to p (see (8.13)).

In terms of the real functions $F_r(x, t)$, $G_r(x, t)$, introduced in (6.13), we have

$$\left.\begin{aligned} F_r(x,t) &= \Phi_r(x,t) + \Phi_{-r}(x,t) = \sum_{s=-\infty}^{\infty} f_{r,s}(t) \exp(s\pi xi/a) \\ &\qquad (r = 0, 1, 2, \dots) \\ G_r(x,t) &= i\{\Phi_r(x,t) - \Phi_{-r}(x,t)\} = \sum_{s=-\infty}^{\infty} g_{r,s}(t) \exp(s\pi xi/a), \\ &\qquad (r = 1, 2, \dots) \end{aligned}\right\}, \quad . \quad . \quad (7.13)$$

where

$$\left.\begin{aligned} f_{r,s}(t) &= \omega_{r,s}(t) + \omega_{-r,s}(t), \qquad (r = 0, 1, 2, \dots;\ s = 0, \pm 1, \dots) \\ g_{r,s}(t) &= i\{\omega_{r,s}(t) - \omega_{-r,s}(t)\}, \quad (r = 1, 2, \dots;\ s = 0, \pm 1, \dots) \end{aligned}\right\}. \quad (7.14)$$

The determination of the motion is reduced to the determination of $\omega_{r,s}(t)$, or, equivalently, of $f_{r,s}(t)$, $g_{r,s}(t)$. We easily derive from (7.10) the equivalent form

$$\left.\begin{aligned} &\frac{b}{2\pi i}\frac{df_{r,s}}{dt} + \tfrac{1}{2}\lambda_s\left(A'_{r,0} + \sum_{p=0}^{\infty} C'_{r,p} K_{p,0,s}\right) f_{0,s} \\ &\quad + \lambda_s \sum_{q=1}^{\infty}\left(A'_{r,q} + \sum_{p=0}^{\infty} C'_{r,p} K_{p,q,s}\right) f_{q,s} \\ &\quad + \lambda_s \sum_{q=1}^{\infty}\left(A''_{r,q} + \frac{b^2}{\pi^2}\frac{1}{q^2+\lambda_s^2} B'_{r,q}\right) g_{q,s} = 0, \\ &\qquad\qquad (r = 0, 1, 2, \dots;\ s = 0, \pm 1, \dots) \\ &\frac{b}{2\pi i}\frac{dg_{r,s}}{dt} + \tfrac{1}{2}\lambda_s\left(A'''_{r,0} + \sum_{p=0}^{\infty} C''_{r,0} K_{p,0,s}\right) f_{0,s} \\ &\quad + \lambda_s \sum_{q=1}^{\infty}\left(A'''_{r,q} + \sum_{p=0}^{\infty} C''_{r,p} K_{p,q,s}\right) f_{q,s} \\ &\quad + \lambda_s \sum_{q=1}^{\infty}\left(A''''_{r,q} + \frac{b^2}{\pi^2}\frac{1}{q^2+\lambda_s^2} B''_{r,q}\right) g_{q,s} = 0 \\ &\qquad\qquad (r = 1, 2, \dots;\ s = 0, \pm 1 \dots) \end{aligned}\right\}, \quad (7.15)$$

where the A's, B's and C's are as given in (6.17) and

$$K_{p,q,s} = \frac{16b^2}{\pi^3}\frac{(-1)^{q+p}(2p+1)}{\{(2p+1)^2 + 4\lambda_s^2\}\{(2p+1)^2 - 4q^2\}}. \quad . \quad . \quad . \quad (7.16)$$

If u_0 is an odd function of y, then, by (6.21), the above equations reduce to

$$\left.\begin{aligned} &\frac{b}{2\pi i}\frac{df_{r,s}}{dt} + \lambda_s \sum_{q=1}^{\infty}\left(A''_{r,q} + \frac{b^2}{\pi^2}\frac{1}{q^2+\lambda_s^2} B'_{r,q}\right) g_{q,s} = 0, \\ &\qquad\qquad (r = 0, 1, 2, \dots;\ s = 0, \pm 1, \dots) \\ &\frac{b}{2\pi i}\frac{dg_{r,s}}{dt} + \tfrac{1}{2}\lambda_s\left(A'''_{r,0} + \sum_{p=0}^{\infty} C''_{r,0} K_{p,0,s}\right) f_{0,s} \\ &\quad + \lambda_r \sum_{q=1}^{\infty}\left(A'''_{r,q} + \sum_{p=0}^{\infty} C''_{r,p} K_{p,q,s}\right) f_{q,s} = 0, \\ &\qquad\qquad (r = 1, 2, \dots;\ s = 0, \pm 1, \dots) \end{aligned}\right\}. \quad (7.17)$$

If u_0 *is an even function of* y, then, by (6.23), we have

$$\left.\begin{aligned} &\frac{b}{2\pi i}\frac{df_{r,s}}{dt}+\tfrac{1}{2}\lambda_s\Big(A'_{r,0}+\sum_{p=0}^{\infty}C'_{r,p}K_{p,0,s}\Big)f_{0,s} \\ &\qquad +\lambda_s\sum_{q=1}^{\infty}\Big(A'_{r,q}+\sum_{p=0}^{\infty}C'_{r,p}K_{p,q,s}\Big)f_{q,s}=0, \\ &\qquad\qquad (r=0,\,1,\,2,\,\dots;\ s=0,\,\pm 1,\,\dots) \\ &\frac{b}{2\pi i}\frac{dg_{r,s}}{dt}+\lambda_s\sum_{q=1}^{\infty}\Big(A''''_{r,q}+\frac{b^2}{\pi^2}\frac{1}{q^2+\lambda_s^2}B''_{r,q}\Big)g_{q,s}=0, \\ &\qquad\qquad (r=1,\,2,\,\dots;\ s=0,\,\pm 1,\,\dots) \end{aligned}\right\}. \quad (7.18)$$

As we already know, there is a separation of the f- and g-functions here. The result of a symmetrical disturbance of a symmetrical velocity-profile is to be found from the g-equation of (7.18), since ω' will be an odd function of y and we shall have $f_{r,s}(t)=0$. In the case where u_0 is odd, the f-functions may be separated from the g-functions by differentiating the equations.

Returning to the general case, for which (7.10) or (7.15) hold, we note that if $s=0$, we have

$$\frac{d\omega_{r,0}}{dt}=0, \qquad (r=0,\,\pm 1,\,\dots).$$

Consequently

$$\omega_{r,0}(t)=\text{const.}=\alpha_{r,0} \qquad (r=0,\,\pm 1,\,\dots), \quad . \ . \ . \ . \quad (7.19)$$

and hence, by (7.14),

$$f_{r,0}(t)+f_{-r,0}(t)=\text{const.}=\alpha_{r,0}+\alpha_{-r,0}, \qquad (r=0,\,1,\,2,\,\dots). \quad . \quad (7.20)$$

On account of (7.1*a*), it is easily seen that

$$\omega_{r,s}(t)=0, \qquad |s|>S. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (7.21)$$

8—*Solution of the system of equations by power series in t*

The fundamental equations (7.10) may be written

$$\frac{1}{\lambda_s}\frac{d\omega_{r,s}}{d\tau}=\sum_{q=-\infty}^{\infty}N_{r,q,s}\,\omega_{q,s} \qquad (r=0,\,\pm 1,\,\dots;\ s=\pm 1,\,\pm 2,\,\dots), \quad . \quad (8.1)$$

where

$$\tau=-i\pi t/b. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8.2)$$

It is not necessary to consider $s=0$, since that has been dealt with in (7.19).

We observe at once that there is no "interaction" between the different values of s: the determination of the set of quantities

$$\omega_{r,s}(t), \qquad (r=0,\,\pm 1,\,\dots),$$

for one assigned s, constitutes in itself a distinct problem. It is quite otherwise with respect to the other subscript, r. A disturbance which is intially represented by a single term of (7.1) will at once commence to spread into other terms, and will demand an infinite series in r for its representation (except when $s = 0$).

Let us now consider the problem of solving the infinite system of differential equations (8.1) with the initial conditions (7.4), for some particular value of s. It is suggested by the form of these equations that we should seek solutions of the type

$$\omega_{r,s}(t) = P_{r,s} \exp(\nu_s \lambda_s \tau), \qquad (r = 0, \pm 1, \ldots), \quad . \quad . \quad . \quad . \quad (8.3)$$

where $P_{r,s}$, ν_s are constants. If we substitute in (8.1) and eliminate $P_{r,s}$, we obtain for ν_s the infinite determinantal equation

$$| \nu_s \delta_{r,q} - N_{r,q,s} |_{(r,\, q = 0,\, \pm 1, \ldots)} = 0, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.4)$$

where $\delta_{r,q}$ is the usual Kronecker delta. If solutions

$$\nu_s^{(p)}, \qquad (p = 0, \pm 1, \ldots),$$

of this equation exist, and if $P_{r,s}{}^{(p)}$ are definite sets of constants satisfying the set of linear equations

$$\nu_s^{(p)} P_{r,s}^{(p)} = \sum_{q=-\infty}^{\infty} N_{r,q,s} P_{q,s}^{(p)}, \qquad (r = 0, \pm 1, \ldots), \quad . \quad . \quad . \quad . \quad (8.5)$$

then

$$\omega_{r,s}(t) = \sum_{p=-\infty}^{\infty} Q_{p,s} P_{r,s}{}^{(p)} \exp(\nu_s^{(p)} \lambda_s \tau), \qquad (r = 0, \pm 1, \ldots), \quad . \quad . \quad (8.6)$$

are formal solutions of (8.1), $Q_{p,s}$ being arbitrary constants. To satisfy the initial conditions (7.4), these constants must be chosen to satisfy

$$\sum_{p=-\infty}^{\infty} Q_{p,s} P_{r,s}{}^{(p)} = \alpha_{r,s}, \qquad (r = 0, \pm 1, \ldots). \quad . \quad . \quad . \quad . \quad . \quad (8.7)$$

Were we dealing with a system with a finite number of degrees of freedom, so that finite and not infinite processes were involved, the above method would be excellent. It does not appear applicable to our problem, however; the equation (8.4) does not appear to have definite roots. Moreover, the term-by-term differentiation of (8.6), which would be an essential part of the argument, is too much to assume without proof of its validity.

Rejecting, then, a solution of (8.1) by a series of exponentials let us investigate the solution by means of power series, which are admirably suited to the type of initial conditions (7.4). We shall develop a systematic method for the determination of the coefficients in the power series, and we shall show that under certain conditions concerning u_0 the power series converge absolutely for all values of the time.

Let us put

$$\omega_{r,s}(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \omega_{r,s}^{(n)} (\lambda_s \tau)^n, \qquad (r=0, \pm 1, \ldots;\ s=\pm 1, \pm 2, \ldots), \quad . \quad (8.8)$$

where $\omega_{r,s}^{(n)}$ are constants to be determined. To satisfy the initial conditions (7.4), we choose

$$\omega_{r,s}^{(o)} = \alpha_{r,s}, \qquad (r=0, \pm 1, \ldots;\ s=\pm 1, \pm 2, \ldots). \quad . \quad . \quad . \quad (8.9)$$

Substituting (8.8) in (8.1), and equating the coefficients of equal powers of τ, we obtain

$$\omega_{r,s}^{(n+1)} = \sum_{q=-\infty}^{\infty} N_{r,q,s}\, \omega_{q,s}^{(n)}, \quad (r=0, \pm 1, \ldots;\ s=\pm 1, \pm 2, \ldots;\ n=0,1,2,\ldots). \quad (8.10)$$

With (8.9), these recurrence formulæ determine formally all the coefficients in (8.8). Two points, however, must be cleared up. First, we must investigate the convergence of the series (8.10) and, secondly, that of the power series (8.8).

With this in view, let us consider the magnitude of $N_{r,q,s}$ as given by (7.12). Let us first effect the summation with respect to p in the last term. Resolving into partial fractions, and making use of the following known results,

$$\left.\begin{aligned}
&\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2 - 4q^2} = 0, \qquad (q=\pm 1, \pm 2, \ldots)\\
&\sum_{p=0}^{\infty} \frac{1}{\{(2p+1)^2 - 4q^2\}^2} = \frac{\pi^2}{64q^2}, \qquad (q=\pm 1, \pm 2, \ldots)\\
&\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}\\
&\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2 + 4\lambda_s^2} = \frac{\pi}{8\lambda_s} \tanh \pi\lambda_s
\end{aligned}\right\}, \qquad (8.11)$$

we easily find

$$\sum_{=0}^{\infty} \frac{(2p+1)^2}{\{(2p+1)^2 + 4\lambda_s^2\}\{(2p+1)^2 - 4q^2\}\{(2p+1)^2 - 4m^2\}}$$

$$= \begin{cases}
-\dfrac{\pi^2}{32} \dfrac{\lambda_s^2}{(q^2+\lambda_s^2)(m^2+\lambda_s^2)} \dfrac{\tanh \pi\lambda_s}{\pi\lambda_s}, & (q, m = 0, \pm 1, \ldots;\ q^2 \neq m^2),\\[2ex]
\dfrac{\pi^2}{32} \dfrac{1}{q^2+\lambda_s^2}\left(\dfrac{1}{2} - \dfrac{\lambda_s^2}{q^2+\lambda_s^2} \dfrac{\tanh \pi\lambda_s}{\pi\lambda_s}\right), & (q, m = \pm 1, \pm 2, \ldots;\ q^2 = m^2),\\[2ex]
\dfrac{\pi^2}{32\lambda_s^2}\left(1 - \dfrac{\tanh \pi\lambda_s}{\pi\lambda_s}\right), & (q = m = 0). \quad . \quad . \quad . \quad (8.12)
\end{cases}$$

When we substitute these expressions in (7.12), we find, after some straightforward reductions,

$$N_{r,q,s} = U_{-r} + \frac{b^2}{\pi^2} \frac{1}{q^2+\lambda_s^2} V_{r-q} - \frac{b^2}{\pi^2} \frac{(-1)^q \lambda_s^2}{q^2+\lambda_s^2} \frac{\tanh \pi\lambda_s}{\pi\lambda_s} \sum_{m=-\infty}^{\infty} \frac{(-1)^m V_{r+m}}{m^2+\lambda_s^2}, \quad (8.13)$$

$$(r, q = 0, \pm 1, \ldots;\ s = \pm 1, \pm 2, \ldots).$$

Now, since V_r are the coefficients of the Fourier series for d^2u_0/dy^2, we shall have

$$|V_0| < K_1, \quad |V_r| < K_1/|r|, \quad (r = \pm 1, \pm 2, \ldots), \quad \ldots \quad (8.14)$$

where K_1 is some positive constant. Hence, if we put

$$S_{r,s} = \sum_{m=-\infty}^{\infty} \frac{(-1)^m V_{r+m}}{m^2 + \lambda_s^2}, \quad (r = 0, \pm 1, \ldots; \; s = \pm 1, \pm 2, \ldots), \quad (8.15)$$

we have

$$|S_{r,s}| < K_1 \left\{ \frac{1}{r^2 + \lambda_s^2} + \sum_{m=-\infty}^{\infty} (m \neq -r) \frac{1}{|r+m| (m^2 + \lambda_s^2)} \right\}. \quad (8.16)$$

Since the expression on the right is unchanged on changing the sign of r, we may assume r positive. Then, for $r > 0$,

$$\begin{aligned} |S_{r,s}| &< K_1 \left\{ \frac{1}{r^2} + \frac{1}{\lambda_s^2 r} + \left(\sum_{m=-\infty}^{-r-1} + \sum_{m=-r+1}^{-1} + \sum_{m=1}^{\infty} \right) \frac{1}{|r+m|\, m^2} \right\} \\ &< K_1 \left\{ \frac{1}{r^2} + \frac{1}{\lambda_s^2 r} + \left(\sum_{m=-\infty}^{-1} + \sum_{m=1}^{r-1} + \sum_{m=r+1}^{\infty} \right) \frac{1}{|m| (m-r)^2} \right\} \\ &< K_1 \left\{ \frac{1}{r^2} + \frac{1}{\lambda_s^2 r} + \sum_{m=1}^{\infty} \frac{1}{m(m+r)^2} + \sum_{m=1}^{\infty} (m \neq r) \frac{1}{m(m-r)^2} \right\}. \quad (8.17) \end{aligned}$$

Now, since $(m+r)^2 > 2mr$,

$$\sum_{m=1}^{\infty} \frac{1}{m(m+r)^2} < \frac{1}{2r} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{12r}. \quad \ldots \quad (8.18)$$

Also

$$\sum_{m=1}^{\infty} (m \neq r) \frac{1}{m(m-r)^2} = \frac{1}{r} \sum_{m=1}^{\infty} (m \neq r) \left\{ \frac{1}{(m-r)^2} - \frac{1}{m(m-r)} \right\}. \quad (8.19)$$

But

$$\sum_{m=1}^{\infty} (m \neq r) \frac{1}{(m-r)^2} < 2 \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{3}, \quad \ldots \quad (8.20)$$

and

$$\sum_{m=1}^{\infty} (m \neq r) \frac{1}{m(m-r)} = \frac{1}{r^2} - \frac{1}{r} \sum_{m=1}^{r-1} \frac{1}{m}, \quad \ldots \quad (8.21)$$

which is bounded for positive values of r, being of the order of $(\log r)/r$. Thus the series on the left of (8.19) is of the order of $1/r$, and hence, since $\lambda_s^2 \geqslant b^2/a^2$, there exists a positive constant K_2 such that

$$\left.\begin{aligned} |S_{0,s}| &\equiv \left| \sum_{m=-\infty}^{\infty} \frac{(-1)^m V_m}{m^2 + \lambda_s^2} \right| < K_2 \\ |S_{r,s}| &\equiv \left| \sum_{m=-\infty}^{\infty} \frac{(-1)^m V_{m+r}}{m^2 + \lambda_s^2} \right| < \frac{K_2}{|r|}, \quad (r, s = \pm 1, \pm 2, \ldots) \end{aligned}\right\} \quad (8.22)$$

We have already made the legitimate assumption in (8.14) that the coefficients V_r in the Fourier expansion of d^2u_0/dy^2 are of the order of $1/|r|$. This is true, by RIEMANN's Lemma*, if d^2u_0/dy^2 is of limited total fluctuation ; we would naturally assume this to be the case in the physical problem. We might legitimately make the same assumption regarding the coefficients U_r of the Fourier expansion of u_0, but this does not appear to be enough to ensure the convergence of the series (8.8), (8.10). *We shall now assume that the coefficients* U_r *in the Fourier expansion of* u_0 *are of the order of* $1/r^2$. This assumption is not altogether unnatural from a physical point of view. We have generally

$$U_r = \frac{1}{2b}\int_{-b}^{b} u_0(y) \exp(-r\pi yi/b)\, dy$$

$$= -\frac{(-1)^r}{2r\pi i}\Big[u_0\Big]_{y=-b}^{y=b} + \frac{1}{2r\pi i}\int_{-b}^{b}\frac{du_0}{dy}\exp(-r\pi yi/b)\, dy. \quad . \quad (8.23)$$

The second term is of the order of $1/r^2$. *Hence in order that* U_r *may be of the order of* $1/r^2$ *it is necessary and sufficient that*

$$u_0(b) = u_0(-b), \quad \ldots \quad (8.24)$$

that is, the slipping-velocities of the steady motion on the two walls must be equal. Since this condition is not in general fulfilled for the motion of an inviscid liquid, it seems that our investigation cannot deal with the general velocity-profile. However, the "inviscid liquid" is at best an artificial conception, acquiring its physical significance by approximating to the liquid of small viscosity. Since the condition (8.24) is necessarily fulfilled for a viscous liquid (no matter how small its viscosity), provided that the walls are at rest, the disturbance of the motion of an inviscid liquid whose velocity-profile satisfies (8.24) is certainly of the greatest physical interest.

Under the stated hypothesis, then, we have

$$|U_0| < K_3, \qquad |U_r| < K_3/r^2, \qquad (r = \pm 1, \pm 2, \ldots), \quad . \; . \; . (8.25)$$

for some positive constant K_3. Thus by (8.13) and (8.22) we see that for any assigned value of s, there exists a positive constant K_4 such that

$$\left.\begin{aligned}
&|N_{r,r,s}| < K_4, && (r = 0, \pm 1, \ldots),\\
&|N_{r,0,s}| < \frac{K_4}{|r|}, && (r = \pm 1, \pm 2, \ldots),\\
&|N_{0,q,s}| < \frac{K_4}{|q|}, && (q = \pm 1, \pm 2, \ldots),\\
&|N_{r,q,s}| < K_4\left\{\frac{1}{(q-r)^2} + \frac{1}{|q-r||q|} + \frac{1}{|qr|}\right\}, && (r, q = \pm 1, \pm 2, \ldots;\ q \neq r)
\end{aligned}\right\} . \quad (8.26)$$

From the argument at (8.17) it follows that

$$\sum_{q=-\infty}^{\infty} \begin{matrix}(q \neq 0)\\(q \neq r)\end{matrix} \frac{1}{|q-r|q^2}, \quad \ldots \quad (8.27)$$

* *Cf.* WHITTAKER and WATSON, "Modern Analysis," p. 166.

is of the order of $1/|r|$. Hence there exists a positive constant C such that

$$\left.\begin{aligned} &K_4\left(1+2\sum_{q=1}^{\infty}\frac{1}{q^2}\right) < C, \\ &K_4\left\{\frac{2}{|r|}+\sum_{q=-\infty}^{\infty}\begin{matrix}(q\neq 0)\\(q\neq r)\end{matrix}\left(\frac{2}{|q-r|q^2}+\frac{1}{q^2|r|}\right)\right\} < \frac{C}{|r|}, \qquad (r=\pm 1, \pm 2, \ldots) \end{aligned}\right\} . \quad (8.28)$$

We are now in a position to discuss the convergence of (8.10). Since $\alpha_{r,s}$ are coefficients in the Fourier series (7.1) with respect to y, it follows from (8.9) that for the assigned value of s

$$|\omega_{0,s}^{(0)}| < K, \qquad |\omega_{r,s}^{(0)}| < K/|r|, \qquad (r=\pm 1, \pm 2, \ldots), \quad . \quad . \quad (8.29)$$

where K is some positive constant. The series (8.10) for $n=0$ will then be convergent, and will define $\omega_{r,s}^{(1)}$ such that

$$\left.\begin{aligned} |\omega_{0,s}^{(1)}| &\leqslant \sum_{q=-\infty}^{\infty}|N_{0,q,s}\omega_{q,s}^{(0)}| < KK_4\left(1+2\sum_{q=1}^{\infty}\frac{1}{q^2}\right) < CK, \\ |\omega_{r,s}^{(1)}| &\leqslant \sum_{q=-\infty}^{\infty}|N_{r,q,s}\,\omega_{q,s}^{(0)}| \\ &\leqslant |N_{r,0,s}\,\omega_{0,s}^{(0)}| + |N_{r,r,s}\,\omega_{r,s}^{(0)}| + \sum_{q=-\infty}^{\infty}\begin{matrix}(q\neq 0)\\(q\neq r)\end{matrix}|N_{r,q,s}\,\omega_{q,s}^{(0)}| \\ &< KK_4\left\{\frac{1}{|r|}+\frac{1}{|r|}+\sum_{q=-\infty}^{\infty}\begin{matrix}(q\neq 0)\\(q\neq r)\end{matrix}\left(\frac{1}{(q-r)^2|q|}+\frac{1}{|q-r|\,q^2}+\frac{1}{q^2|r|}\right)\right\} \\ &< \frac{CK}{|r|}, \qquad (r=\pm 1, \pm 2, \ldots) \end{aligned}\right\} . \quad (8.30)$$

When we apply the inequalities thus written to (8.10) with $n=1$, we again obtain convergent series, and we find

$$|\omega_{0,s}^{(2)}| < C^2K, \qquad |\omega_{r,s}^{(2)}| < C^2K/|r|, \qquad (r=\pm 1, \pm 2, \ldots). \quad . \quad . \quad (8.31)$$

Hence, by induction, all the series (8.10) converge absolutely and we have

$$|\omega_{0,s}^{(n)}| < C^nK, \qquad |\omega_{r,s}^{(n)}| < C^nK/|r|, \qquad (r=\pm 1, \pm 2, \ldots;\ n=0, 1, 2, \ldots). \quad . \quad (8.32)$$

Thus the nth term of the power series (8.8) is less in absolute value than

$$\frac{1}{n!}C^nK\,|\lambda_s\tau|^n \quad \text{if} \quad r=0,$$

$$\frac{1}{n!}\frac{C^nK}{|r|}|\lambda_s\tau|^n \quad \text{if} \quad r=\pm 1, \pm 2, \ldots.$$

But these are the nth terms of series which converge for all values of τ. Hence the series (8.8) converge absolutely for all values of τ. Moreover, we have

$$|\omega_{r,s}(t)| < \frac{K}{|r|}\sum_{n=0}^{\infty}\frac{C^n|\lambda_s\tau|^n}{n!}, \qquad (r=\pm 1, \pm 2, \ldots). \quad . \quad . \quad . \quad (8.33)$$

This shows that $\omega_{r,s}(t)$ is of the order of $1/|r|$, but it does not necessarily follow that the series

$$\sum_{r=-\infty}^{\infty} \omega_{r,s}(t) \exp(r\pi yi/b),$$

converges, although we have no reason to suppose that it diverges. We shall assume convergence, leaving the argument incomplete in this respect.

Let us now sum up the results of this discussion.

THEOREM VIII—*For any assigned value of s the equations* (7.10) *possess solutions* $\omega_{r,s}(t)$ *in the form of power series in t* (8.8), *which converge for all values of t, and satisfy the assigned initial conditions* (7.4), *provided that the velocity of the undisturbed motion is the same on both walls.** *The coefficients in the power series are determined by the recurrence formulæ* (8.10) *The vorticity of the disturbance at time t is given by* (7.9).

As regards the analytical validity of the processes, we are to remember that, on account of (7.21), (7.9) is not in reality an infinite series with respect to s. Consequently there is no question of the validity of the term-by-term differentiation effected in (7.7). It will be noted that we have never found it necessary to differentiate (7.9) term-by-term with respect to y: that would, in general, not be valid. It is hoped, therefore that Theorems VII and VIII contain the correct solution to the problem of the small periodic disturbance of the steady flow of an inviscid liquid between parallel planes, the only restrictions being

(i) the velocities on the two walls in the steady motion are equal;

(ii) the initial disturbance is represented by a finite number of sinusoidal terms as far as its dependence on x is concerned.

The theory is valid even though d^2u_0/dy^2 should have a finite number of discontinuities, du_0/dy being continuous.

9—*Calculation of the coefficients in the power series*

The calculation of the coefficients in the power series (8.8) by means of the recurrence formulæ (8.9), (8.10) suffers from the defect that a separate calculation is required for the study of each disturbance of a given velocity-profile. We shall now remedy this defect, and show how one calculation will suffice for all disturbances, the value of $|s|$ (*i.e.*, the wave-length of the disturbance with respect to x) being assigned.

* This condition is, of course, fulfilled if the velocity-profile is represented by a finite number of terms of the series $\sum_{r=-\infty}^{\infty} U_r \exp(r\pi yi/b)$.

It follows from the recurrence formulæ (8.9), (8.10) that $\omega^{(n)}_{r,s}$ will be linear functions of

$$\ldots, \alpha_{-1,s}, \alpha_{0,s}, \alpha_{1,s}, \ldots.$$

Let us put

$$\omega^{(n)}_{r,s} = \sum_{p=-\infty}^{\infty} \mathrm{K}^{(n)}_{r,p,s}\,\alpha_{p,s}, \qquad (r = 0, \pm 1, \ldots;\ s = \pm 1, \pm 2, \ldots;\ n = 0, 1, 2, \ldots), \quad (9.1)$$

the quantities $\mathrm{K}^{(n)}_{r,p,s}$ being independent of the α's. Substituting in (8.10), and regarding the α's as arbitrary, we obtain

$$\mathrm{K}^{(n+1)}_{r,p,s} = \sum_{q=-\infty}^{\infty} \mathrm{N}_{r,q,s}\,\mathrm{K}^{(n)}_{q,p,s}, \quad (r = 0, \pm 1, \ldots;\ s = \pm 1, \pm 2, \ldots;\ n = 0, 1, 2, \ldots). \quad (9.2)$$

Thus if we denote by $\mathbf{K}^{(n)}_s$ the matrix whose (r, p) element is

$$\mathrm{K}^{(n)}_{r,p,s},$$

and by $\mathbf{N}_s$ the matrix whose (r, p) element is

$$\mathrm{N}_{r,p,s},$$

then (9.2) reads

$$\mathbf{K}^{(n+1)}_s = \mathbf{N}_s \,.\, \mathbf{K}^{(n)}_s, \quad (9.3)$$

and hence we obtain

$$\mathbf{K}^{(n)}_s = \mathbf{N}^n_s \,.\, \mathbf{K}^{(0)}_s, \quad (9.4)$$

where $\mathbf{N}^n_s$ is the nth power of the matrix $\mathbf{N}_s$. But putting $n = 0$ in (9.1) and remembering that $\omega^{(0)}_{r,s} = \alpha_{r,s}$ by (8.9), we have

$$\mathbf{K}^{(0)}_s = \mathbf{1}. \quad (9.5)$$

Thus (9.4) reads

$$\mathbf{K}^{(n)}_s = \mathbf{N}^n_s. \quad (9.6)$$

The problem of calculating the coefficients $\omega^{(n)}_{r,s}$ *thus reduces to the calculation of the nth power of the matrix* $\mathbf{N}_s$. To see whether this calculation is possible, we revert to (8.26), which shows the order of magnitude of the elements of $\mathbf{N}_s$; we shall write, however, as is obviously legitimate,

$$\left.\begin{aligned}
&|\mathrm{N}_{r,r,s}| < \mathrm{K}_4, && (r = 0, \pm 1, \ldots)\\
&|\mathrm{N}_{r,0,s}| < \frac{\mathrm{K}_4}{|r|}, && (r = \pm 1, \pm 2 \ldots)\\
&|\mathrm{N}_{0,q,s}| < \frac{\mathrm{K}_4}{|q|}, && (q = \pm 1, \pm 2, \ldots)\\
&|\mathrm{N}_{r,q,s}| < \mathrm{K}_4\left\{\frac{1}{(q-r)^2} + \frac{1}{|q-r|\,|r|} + \frac{1}{|q-r|\,|q|} + \frac{1}{|qr|}\right\},\\
& && (r, q = \pm 1, \pm 2, \ldots;\ r \neq q)
\end{aligned}\right\} \quad (9.7)$$

Now when we form the matrix

$$\sum_{p=-\infty}^{\infty} N_{r,p,s} N_{p,q,s}, \qquad (9.8)$$

it is easy to see that the series converges and that the moduli of the elements of the matrix so formed satisfy inequalities of the form (9.7) but with a different constant: we make use of the result, established in §8, that

$$\sum_{m=-\infty}^{\infty} \begin{pmatrix} m \neq 0 \\ m \neq -r \end{pmatrix} \frac{1}{m^2 |m+r|}, \quad \text{or} \quad \sum_{m=-\infty}^{\infty} \begin{pmatrix} m \neq 0 \\ m \neq r \end{pmatrix} \frac{1}{(m-r)^2 |m|},$$

is of the order of $1/|r|$. Hence it follows by induction that all the matrices N_s^n exist, although it is by no means to be assumed that their elements remain finite as n tends to infinity.

Let us now consider the case of *the parabolic velocity-profile*

$$u_0(y) = k(b^2 - y^2). \qquad (9.9)$$

We have then

$$U_r = \frac{1}{2b}\int_{-b}^{b} k(b^2 - y^2) \exp(-r\pi yi/b)\, dy$$

$$= \begin{cases} \frac{2}{3}kb^2 & \text{if } r = 0, \\ -(-1)^r \dfrac{2kb^2}{\pi^2 r^2} & \text{if } r \neq 0; \end{cases} \qquad (9.10)$$

$$V_r = \frac{1}{2b}\int_{-b}^{b} \frac{d^2u_0}{dy^2} \exp(-r\pi yi/b)\, dy$$

$$= \begin{cases} -2k & \text{if } r = 0, \\ 0 & \text{if } r \neq 0. \end{cases} \qquad (9.11)$$

Thus, by (8.13),

$$\left.\begin{aligned} N_{r,r,s} &= \frac{2kb^2}{\pi^2}\left\{\frac{\pi^2}{3} - \frac{1}{r^2+\lambda_s^2} + \frac{\lambda_s^2}{(r^2+\lambda_s^2)^2}\frac{\tanh \pi\lambda_s}{\pi\lambda_s}\right\} \\ N_{r,q,s} &= \frac{2kb^2}{\pi^2}(-1)^{r-q}\left\{-\frac{1}{(r-q)^2} + \frac{\lambda_s^2}{(r^2+\lambda_s^2)(q^2+\lambda_s^2)}\frac{\tanh \pi\lambda_s}{\pi\lambda_s}\right\}, \\ &(r, q = 0, \pm 1, \pm 2, \ldots;\ q \neq r) \end{aligned}\right\} \qquad (9.12)$$

Although the calculation of the elements of N_s^2, N_s^3, ... presents no difficulty, the elements increase in complexity of expression. We shall not trouble to write down analytical expressions for them. The task of preparing arithmetical tables of N_s^2, N_s^3, ... is not an impossible one. The factor $2kb^2/\pi^2$ does not enter into the calculation: it merely appears as a factor $(2kb^2/\pi^2)^n$ in N_s^n. The rest of the calculation is purely arithmetical, when the value of λ_s has been assigned. We may note that for the limiting cases $\lambda_s \to 0$, $\lambda_s \to \infty$ (corresponding respectively to infinitely long and infinitely short waves of disturbance) we have

$$\lim_{\lambda_s \to \infty} \left\{ \begin{array}{l} N_{0,0,s} = 0 \\ N_{r,0,s} = N_{0,r,s} = 0 \\ N_{r,r,s} = \dfrac{2kb^2}{\pi^2}\left(\dfrac{\pi^2}{3} - \dfrac{1}{r^2}\right) \\ N_{r,q,s} = -\dfrac{2kb^2}{\pi^2}\dfrac{(-1)^{r-q}}{(r-q)^2} \end{array} \right\} \quad (r, q = \pm 1, \pm 2, \ldots;\ r \neq q); \tag{9.13}$$

$$\lim_{\lambda_s \to \infty} \left\{ \begin{array}{l} N_{r,r,s} = \dfrac{2kb^2}{\pi^2} \cdot \dfrac{\pi^2}{3} \\ N_{r,q,s} = -\dfrac{2kb^2}{\pi^2}\dfrac{(-1)^{r-q}}{(r-q)^2} \end{array} \right\} \quad (r, q = 0, \pm 1, \pm 2, \ldots;\ r \neq q). \tag{9.14}$$

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